Information Trade-offs in Dynamic Financial Markets*

Job Market Paper

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Abstract

I develop a model of information acquisition in dynamic financial markets. In equilibrium, prices reflect investors' expectations about the cash flows and the supply of a risky asset. Contrary to static models, supply has a significant informational role in markets because it predicts capital gains. Investors decide whether to obtain superior information about dividends at a cost, which also enables them to learn information about supply. Investors who decide to be uninformed learn about dividends and supply from prices. I show that as more informed investors enter the economy, prices become more informative about dividends but less informative about supply. This trade-off creates complementarities in information acquisition. As a result, the information market has multiple equilibria, each with different implications for the financial market.

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1 Introduction

One of the central questions of finance is how information gets incorporated into asset prices. This topic has occupied the minds of economists at least as far back as Hayek (1945), who points out that information enters prices through the trades of informed investors. But exactly what kind of information do prices incorporate? In static, one-period descriptions of asset markets, information is about the liquidating cash flows of risky assets. In dynamic markets, however, investors buy assets to enjoy a stream of dividends and at the same time they are concerned with future price fluctuations. Therefore, prices convey not only what informed investors know about future cash flows, but also what they know about future prices.

By taking this feature of dynamic markets into account we are able to gain new insights into information acquisition. From the perspective of uninformed agents, current prices are a noisy signal of dividend information because they depend on unknown information about future prices. At the same time, current prices are a noisy signal of information about future prices because they depend on unknown dividend information. In other words, prices are non-fully revealing signals of either type of information. This stands in contrast to static markets, where the component of prices that prevents them from fully revealing dividend information does not convey useful information.

I study endogenous information in dynamic markets by extending the static economy of Grossman and Stiglitz (1980). I show that acquiring information in static markets and acquiring information in dynamic markets are qualitatively very different. In the one-period version of the economy the value of information is always decreasing in the number of informed agents. As more informed agents enter the economy, having private information is less valuable because there is more dividend information available publicly through prices. On the contrary, in the dynamic version of the economy, the value of private information can be increasing in the number of informed agents. This distinction arises because higher incidences of informed agents have different implications for the two types of information embedded in prices of dynamic markets.

In order to analyze information acquisition tractably, I model information about future prices as a stochastic stock supply process.[1] I first consider the case where the persistence of supply is low because in this case supply captures information about short-term capital gains. If, for instance, supply is independent over time, changes in supply are predictable by levels of supply. Moreover, because prices depend on supply, levels of supply predict how prices change in the near future. This predictability makes supply a vehicle of valuable information in dynamic financial markets.

For example, suppose that in between dividend payouts liquidity traders enter and leave the market for reasons exogenous to the stock, exposing all other market participants to fluctuations

[1] This is in the spirit of Campbell and Kyle (1993), who make returns predictable by stochastic supply levels. It is also a natural extension of the assumptions of Grossman and Stiglitz (1980) to a dynamic setting.

in supply.[2] Using the predictability of supply fluctuations by supply levels, long-term market participants can take positions in the stock so as to generate capital gains at the expense of the liquidity traders. This aspect of supply information as a source of profits means that investors are willing to pay for information on supply levels.

In order to simplify the analysis, I assume that investors make information decisions once and for all before trading begins. They decide whether to obtain information about the dividend at a fixed cost or remain uninformed for free, knowing that they will be observing prices in the financial market. Prices, however, depend on dividend information and supply. Thus, investors who purchase superior dividend information also have superior supply information because they observe prices. Investors who do not purchase information have to estimate both dividend and supply information from prices. Furthermore, the exact amount that investors are willing to pay for superior information depends on how many other investors become informed. This is because the number of informed agents determines the informativeness of prices about dividends and supply.

In fact, for agents who decide to be uninformed, the number of informed agents presents a trade-off in how much information they can extract from prices. As I have argued above, prices are a noisy version of dividend information because they depend on unknown supply. Similarly, prices are a noisy version of supply because they depend on unknown dividend information. Thus the total amount of information contained in prices is a weighted average of dividend information and supply information. The weight of each type of information is determined in equilibrium as a function of the number of informed agents. Most importantly, if the weight of one type of information increases, then the weight of the other type of information decreases. Now consider what happens as more informed agents enter the economy. As in classic information acquisition, prices become less noisy versions of dividends because there is more dividend information available. This implies that the weight of dividend information in the information content of prices increases. But it also implies that the weight of supply information decreases, and therefore prices become noisier versions of supply.[3]

That is, higher numbers of informed agents make prices more informative about dividends but less informative about supply. This information trade-off has great implications for how much agents are willing to pay for information. On the one hand, more informed agents make private information *less* valuable because dividend information becomes easier to extract from prices. On the other hand, more informed agents also make private information *more* valuable because private information contains superior supply information, which becomes harder to extract from prices. This is important because supply captures information about short-term capital gains.

When there are few informed agents in the economy, most of the profits of agents who opt to be uninformed come from having accurate predictions of capital gains. As the number of informed agents increases, uninformed agents lose accuracy of supply information and gain ac-

[2]For a literature summary and critique of noise traders and liquidity traders, see Dow and Gorton (2008).
[3]See section 5.1 for a detailed analysis.

curacy of dividend information, but they must also face more traders with superior information. This implies that they lose benefits from supply information faster than they receive benefits from improved dividend information. As a result, the ex-ante willingness of agents to pay for information is increasing in the number of informed agents. Namely, other agents becoming informed is a complement for uniformed agents acquiring information directly.

As the number of informed agents keeps increasing, agents who decide to be uninformed give up the ability to predict capital gains because they get compensated by knowing more precise dividend information. Moreover, higher numbers of informed agents implies that agents who decide to be informed face more competition for information rents. As a result, for large numbers of informed agents the ex-ante willingness of agents to pay for information is decreasing in the number of informed agents. In this case, other agents becoming informed is a substitute for uniformed agents acquiring information directly.

The information complementarity for small numbers of informed agents combined with the information substitutability for large numbers of informed agents produces a value of information that is non-monotonic in the number of informed agents. Hence, in an economy where asset prices and levels of information are determined jointly there can be more than one equilibrium. I construct examples with two equilibria of equal informational value: one with a low number of informed agents and another with a high number of informed agents. Uninformed agents estimate dividends more accurately in the latter equilibrium than in the former equilibrium, and they estimate supply more accurately in the former equilibrium than in the latter equilibrium. The two equilibria correspond to time-series of prices with potentially very different properties.

So far I have assumed that the level of supply available in every period is independent of its past. I generalize the supply process by making it persistent and I show that supply persistence plays an important role in whether complementarities emerge. An example of persistent supply is the following. Consider a trader who loses his job and is thereby forced to sell some of his stock. How much he supplies to the market will exhibit persistence over time: every day that he remains unemployed he is forced to sell some more of his stock.

This generalized supply assumption allows me to examine how information acquisition works in asset-pricing environments with capital gains that are predictable by mean-reverting supply. Examples of such economies are the smart-money model of Campbell and Kyle (1993) and the asymmetric information model of Wang (1993). Here, how much the level of supply predicts supply fluctuations depends on the persistence of supply. If supply is more persistent, the level of supply predicts supply fluctuations less, and thus supply has smaller informational value. Because information complementarities come only from supply information effects, more persistent supply produces less pronounced information complementarities. The complementarities disappear completely in the limit case of when supply follows a random walk. But this case is economically troublesome because it implies that prices are not stationary even if economic fundamentals are stationary. Therefore, in dynamic markets, information complementarities are not only more economically interesting than information substitutabilities, they are also more plausible.

The model produces two empirical predictions. Firstly, when complementarities are present, prices can be fragile in perturbations of information costs. When the persistence of supply is low, the value of information as a function of the number of informed agents has the shape of an inverted U for large parts of the parameter space. Suppose that the cost of acquiring information corresponds to a value of information very near the peak of the inverted U. In this case there are two equilibrium incidences of informed agents, one to the left of the peak and one to the right of the peak. Further suppose that the economy is at the equilibrium to the left of the peak. Then a small perturbation in the cost can move the equilibrium to the right of the peak. Because the equilibrium number of informed agents is a parameter in prices, this results in shifts in the variability of prices.

Secondly, in the absence of dividend news, extremely persistent supply inhibits uninformed investors' learning about dividends. Initial price observations help the uninformed update their prior beliefs about dividends, but subsequent prices contribute nothing to learning. This contradicts the casual intuition that the value of fundamentals would be gradually revealed through prices by informed agents trading their information away. Moreover, it is particularly puzzling because what the informed know about dividends does not change. The explanation is that extremely persistent supply makes intertemporal information completely unrelated to dividends. If there are no direct news about dividends, investors use changes in prices in order to learn news indirectly.[4] I show that when supply follows a random walk, equilibrium price changes do not convey any information about dividends. On the contrary, when the persistence of supply is low, price changes do reveal some dividend information. In this case, longer price series correspond to more precisely known dividend information.

In this study the dependence of prices on supply plays a dual role. It conveys non-dividend information about future prices and it makes prices noisy signals of dividend information. Using economic variables other than supply to replicate this role will not change the informativeness trade-off and the main result. Several examples of such interpretations can be found in the literature. Diamond and Verrecchia (1981) model price noise as unknown aggregate supply of stock, where the private stock endowment of each agent is a signal of aggregate supply. Wang (1994) assumes that price noise arises from private investment opportunities available only to informed investors. Finally, Vives (2008) section 4.4.1 gives an environment where the full revelation of information is prevented by the presence of agents that hedge against non-traded goods.

The studies closest to this paper are Grossman and Stiglitz (1980), that considers a single-period financial market, and Wang (1993), that derives a dynamic equilibrium with an exogenous number of informed agents. In related work, Grundy and McNichols (1989) study the volume of trade in a multiperiod noisy rational expectations model. Holden and Subrahmanyam (2002) generate momentum in asset returns in a model where agents can trade before they acquire information. Manela (2011) calibrates a two-trading period Grossman-Stiglitz economy in

[4]For example, the price change between yesterday and today is a channel of today's news because it conveys what was not known yesterday but becomes known today.

order to study how exogenous rates of information dissemination through media affect returns in the market for pharmaceutical drugs. These papers build on the concept of noisy rational expectations introduced by Lucas (1972) and applied to information flows by Green (1973), Grossman (1976, 1978), Kihlstrom and Mirman (1975) and Grossman, Kihlstrom, and Mirman (1977). In this paper I focus on how the information market works when the financial market is intertemporal. I show that when price noise is not persistent there are complementarities in information acquisition. Moreover, I only consider informed agents that hold the same piece of information. Papers that study the aggregation of diverse information and its effects on information acquisition include Hellwig (1980), Diamond and Verrecchia (1981), Verrecchia (1982) and others.

This paper also relates to a growing literature about complementarities in information acquisition. There are several mechanisms that produce information complementarities in a static environment. They include correlated fundamentals and supply (Barlevy and Veronesi, 2000, 2008), fixed costs in the information production sector as in Veldkamp (2006), supply signals (Ganguli and Yang, 2009; Manzano and Vives, 2011), non-normal returns (Breon-Drish, 2010), ambiguity aversion (Mele and Sangiorgi, 2011), information acquisition in segmented markets (Goldstein, Li, and Yang, 2011), information acquisition with diverse information (Goldstein and Yang, 2011) and others. What is new with respect to this literature is that complementarities obtain in a CARA-normal environment by assuming that financial markets are dynamic. This natural assumption produces a supply information channel which works in opposition to the dividends information channel.

The next section describes the setup of the model. Section 3 describes the equilibrium in the financial market assuming an equilibrium in the information market. Section 4 describes the overall equilibrium in the information market. Section 5 discusses and analyzes the information trade-off, complementarities and further results. Section 6 considers how the information market equilibrium works in a continuous-time version of the model and argues that the two-period environment is enough to capture the economic forces at work. Finally, section 7 summarizes the paper and outlines avenues of further research.

2 The Model

There is a continuum of ex-ante identical agents of total mass one. Each agent has constant absolute risk aversion preferences with coefficient δ. Everyone has initial wealth W_0 which he can invest in a safe storage technology with constant net return normalized to zero and in a risky stock. There are two trading periods, $t = 1$ and $t = 2$, during which agents trade the stock but do not consume. This is followed by a consumption-only period, $t = 3$. The stock pays off a risky dividend D_3 only in the consumption period. It is known that the dividend is made up of two parts,

$$D_3 = \tilde{\mu} + \tilde{\zeta},$$

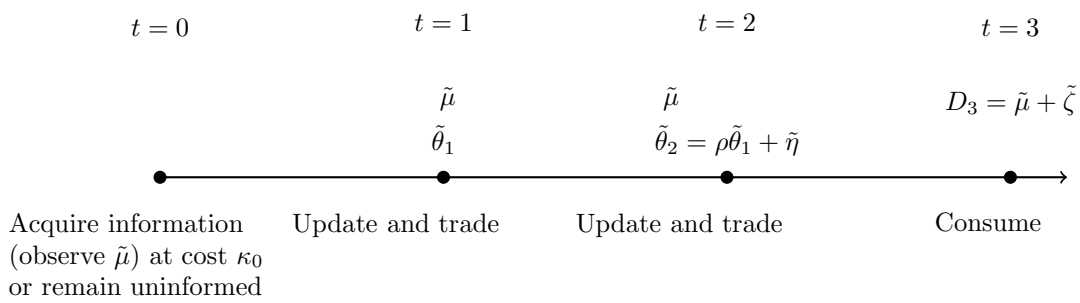


Figure 1: The sequence of events in the model.

where $\tilde{\mu} \sim \mathcal{N}(0, \sigma_\mu^2)$ and $\tilde{\zeta} \sim \mathcal{N}(0, \sigma_\zeta^2)$ with $\tilde{\mu}$ independent of $\tilde{\zeta}$. Before the two trading periods there is an information acquisition period, $t = 0$, during which every agent can pay κ_0 so as to observe the value of $\tilde{\mu}$ right before trading starts at $t = 1$. The random variable $\tilde{\zeta}$ captures the residual dividend uncertainty faced by informed agents. An agent might also decide not to pay κ_0 at the information acquisition stage and thereby remain uninformed for the duration of trading. Thus for an uninformed agent D_3 is a random variable over which he has the prior $D_3 \sim \mathcal{N}(0, \sigma_\mu^2 + \sigma_\zeta^2)$.

At $t = 1$ the price is P_1 and at $t = 2$ the price is P_2. Because prices depend on the value of $\tilde{\mu}$, the uninformed agent uses prices in order to update his beliefs about $\tilde{\mu}$. Let $\hat{\mu}_1$ and $\hat{\mu}_2$ be the uninformed agent's estimate of $\tilde{\mu}$ at $t = 1$ and $t = 2$.

If prices contained no more unknowns than just the mean dividend $\tilde{\mu}$, the uninformed would be able to infer the mean dividend perfectly and there would be no environment of asymmetric information. Thus the presence of noise is necessary in prices. As is standard in the literature, I fill this modeling necessity by making the supply of stock at time t, $\tilde{\theta}_t$, stochastic. Furthermore I assume that

$$\tilde{\theta}_2 = \rho\tilde{\theta}_1 + \tilde{\eta}$$

where I take $0 \leq \rho \leq 1$.[5] Here $\tilde{\theta}_1$ and $\tilde{\eta}$ are independent of each other and of $\tilde{\mu}$ and $\tilde{\zeta}$. When $\rho = 0$ the supply is independent over time. This has the interpretation of noise traders arriving each period in the financial market, as in Kyle (1985). When $\rho > 0$ some noise traders that entered the market at $t = 1$ remain in the market for $t = 2$. Positive ρ therefore measures the persistence of liquidity shocks faced by the noise traders. The priors over supply are $\tilde{\theta}_1 \sim \mathcal{N}(0, \sigma_\theta^2)$ and $\tilde{\eta} \sim \mathcal{N}(0, \sigma_\eta^2)$. If an agent decides to become informed he will not only see $\tilde{\mu}$, but also the price at $t = 1$ and $t = 2$. As explained shortly, the informed agents will be able to deduce perfectly the level of supply at each point in time. At the same time, the uninformed agents will be able to estimate also this piece of information by observing prices. Let $\hat{\theta}_1$ be the uninformed agent's estimate of $\tilde{\theta}_1$ at $t = 1$ and let $\hat{\theta}_2$ be his estimate of $\tilde{\theta}_2$ at $t = 2$.

See figure 1 for a graphical depiction of the sequence of events in the model. Each agent an-

[5] This supply assumption is the discrete-time equivalent of the mean-reverting supply assumption of Campbell and Kyle (1993) and Wang (1993).

ticipates rationally the trade-and-update process at the two trading periods. In the information-acquisition period he compares the benefits of being informed versus the cost of giving up κ_0. The benefit of being informed, however, depends on the number of agents that will have decided to be informed, because the presence of informed agents influences the informativeness of prices. Let λ denote the fraction of informed agents. The information market will equilibrate at the λ at which every agent is indifferent between paying to become informed and remaining uninformed for free. It is also possible that information is too cheap, in which case the economy will equilibrate at $\lambda = 1$, or that information is too expensive, in which case the economy will equilibrate at $\lambda = 0$.

3 Financial Market Equilibrium

I construct the equilibrium in the financial market in a conjecture-and-verify approach. The equilibrium concept I use is that of rational expectations, developed by Lucas (1972), Green (1973), Grossman (1976) and Kreps (1977). In particular, I use a discrete-time finite-horizon version of the continuous-time steady-state equilibrium of Wang (1993). Let i denote the informed agents and u denote the uninformed agents.

Definition 1 (Financial Market Equilibrium)**.** A *Financial Market Equilibrium at a fraction λ of informed agents* is a pair of prices $\left(P_1^\lambda, P_2^\lambda\right)$ such that

(a) Agents in group j, $j = i, u$, select their demand in period t based on their information set $\mathcal{F}_t^j$ so as to maximize expected utility.

(b) The prices $\left(P_1^\lambda, P_2^\lambda\right)$ are such that in each period total demand for the stock equals total supply of the stock.

(c) Agents in group j, $j = i, u$, extract their information sets $\mathcal{F}_t^j$ rationally from the history of prices and any other information available to them in period t, for $t = 1, 2$.

At $t = 1$ the value function of the uninformed is $J^u(W_0, P_1^\lambda; \lambda)$ and the value function of the informed is $J^i(W_0, P_1^\lambda, \tilde{\mu}; \lambda)$. For the rest of this section I drop the dependence on λ.

3.1 Prices and Information

I conjecture that prices are linear in state variables for a fixed number λ of informed agents:

$$P_1 = q_\mu \tilde{\mu} + q_\theta \tilde{\theta}_1 + q_{\hat{\mu}} \hat{\mu}_1 + q_{\hat{\theta}} \hat{\theta}_1, \tag{1a}$$

$$P_2 = p_\mu \tilde{\mu} + p_\theta \tilde{\theta}_2 + p_{\hat{\mu}} \hat{\mu}_2 + p_{\hat{\theta}} \hat{\theta}_2. \tag{1b}$$

The only objects in prices that do not depend on λ are $\tilde{\mu}$, $\tilde{\theta}_1$ and $\tilde{\theta}_2$. At $t = 1$ an uninformed agent observes P_1 only and at $t = 2$ he observes P_2 only, but remembers P_1. Thus information

sets of the uninformed are the σ-algebras

$$\mathcal{F}_1^u = \sigma(P_1)$$
$$\mathcal{F}_2^u = \sigma(P_1, P_2).$$

The uninformed estimate mean dividends and supply level given their information at each point in time,

$$t = 1: \ \hat{\mu}_1 = \mathbb{E}\left[\mu|\mathcal{F}_1^u\right], \ \hat{\theta}_1 = \mathbb{E}\left[\theta_1|\mathcal{F}_1^u\right],$$
$$t = 2: \ \hat{\mu}_2 = \mathbb{E}\left[\mu|\mathcal{F}_2^u\right], \ \hat{\theta}_2 = \mathbb{E}\left[\theta_2|\mathcal{F}_2^u\right].$$

These inferences are given as the solution to a Kalman filter problem solved in the appendix. Here I present briefly how this estimation works. The inferences $\hat{\mu}_1$ and $\hat{\theta}_1$ belong to $\mathcal{F}_1^u$ so the uninformed treat them as known. Therefore from (1a) what the uninformed observe when they see P_1 is the price signal

$$y_1 = q_\mu \tilde{\mu} + q_\theta \tilde{\theta}_1.$$

The inferences $\hat{\mu}_1$ and $\hat{\theta}_1$ are linear transformations of this price signal. Similarly at $t = 2$ what the uninformed observe when they see P_2 is the price signal

$$y_2 = p_\mu \tilde{\mu} + p_\theta \tilde{\theta}_2,$$

so the inferences $\hat{\mu}_2$ and $\hat{\theta}_2$ are linear combinations of this price signal and y_1. This also means that $\hat{\mu}_2$ and $\hat{\theta}_2$ are linear combinations of $\tilde{\mu}$, $\tilde{\theta}_1$ and $\tilde{\theta}_2$. As a result the price representation (1) is equivalent to

$$P_1 = q'_\mu \tilde{\mu} + q'_\theta \tilde{\theta}_1, \tag{2a}$$
$$P_2 = p'_\mu \tilde{\mu} + p'_{\theta_2} \tilde{\theta}_2 + p'_{\theta_1} \tilde{\theta}_1. \tag{2b}$$

where q'_μ and q'_θ are linear combinations of the price coefficients in (1a) and p'_μ, p'_{θ_2} and p'_{θ_1} are linear combinations of the price coefficients in (1b).[6]

Moreover, for uninformed agents observing the price signals y_1 and y_2 is equivalent to observing

$$\frac{1}{q_\theta} y_1 = q_{\mu\theta} \tilde{\mu} + \tilde{\theta}_1$$

and

$$\frac{1}{p_\theta} y_2 = p_{\mu\theta} \tilde{\mu} + \tilde{\theta}_2,$$

where $q_{\mu\theta}$ denotes the ratio q_μ / q_θ and $p_{\mu\theta}$ denotes the ratio p_μ / p_θ. Therefore prices are noisy

[6] The price function (2a) is the same as that in Grossman and Stiglitz (1980) and (2b) is the extension of their price function to the second period. The representation (1) is motivated by Wang (1993) and is easier to work with in this setup.

signals of dividends where the noise is the supply and $q_{\mu\theta}$ and $p_{\mu\theta}$ are *sensitivities of price information to dividend information.* Moreover, because at $t = 2$ the price signal at $t = 1$ has already been seen the new information content of y_2 is

$$\frac{1}{p_\theta} y_2 - \frac{\rho}{q_\theta} y_1 = (p_{\mu\theta} - \rho q_{\mu\theta})\, \tilde{\mu} + \tilde{\eta}.$$

This says that $p_{\mu\theta} - \rho q_{\mu\theta}$ is the *sensitivity of price information to dividend information adjusted for learning.*

The informed agents also observe the price signals y_1 and y_2. But because they know $\tilde{\mu}$, they effectively observe $\tilde{\theta}_1$ and $\tilde{\theta}_2$. Thus the information sets of the informed are the σ-algebras

$$\begin{aligned} \mathcal{F}_1^i &= \sigma(\tilde{\mu}, P_1) = \sigma(\tilde{\mu}, \tilde{\theta}_1) \\ \mathcal{F}_2^i &= \sigma(\tilde{\mu}, P_1, P_2) = \sigma(\tilde{\mu}, \tilde{\theta}_1, \tilde{\theta}_2). \end{aligned}$$

The information market is open only at time zero, at which time all agents are identically uninformed. Everyone's initial information set $\mathcal{F}_0^u$ is fully characterized by the priors as described above.

3.1.1 The Difference From Static Financial Markets

There are three returns in the model, $D_3 - P_2$, $D_3 - P_1$ and $P_2 - P_1$. In a static market with trading at $t = 2$ the only driver of the value of information would be the informativeness of P_2 about $D_3 - P_2$. The dynamic market has an additional trading period at $t = 1$, at which time the agents are concerned with how well they know $D_3 - P_1$ and $P_2 - P_1$. The informativeness of P_1 about $D_3 - P_1$ is similar to that in a static market with trading at $t = 1$. The main difference between my model and a static market is the informativeness of P_1 about returns between $t = 1$ and $t = 2$,

$$P_2 - P_1 = \left(p'_\mu - q'_\mu\right) \tilde{\mu} + \left(\rho p'_{\theta_2} + p'_{\theta_1} - q'_\theta\right) \tilde{\theta}_1 + p'_{\theta_2} \tilde{\eta}.$$

Because there are no intermediate dividends the returns are capital gains. There are two channels through which price communicates information about the capital gain, $\tilde{\mu}$ and $\tilde{\theta}_1$. The quantity $\rho p'_{\theta_2} + p'_{\theta_1} - q'_\theta$ links $P_2 - P_1$ to supply. The estimation of $\tilde{\theta}_1$ improves when prices are more informative about supply. Prices are more informative about supply when they load more on $\tilde{\theta}_1$. But that means that they are noisier signals of dividends. That is, although the supply channel conveys information about the capital gain it plays the role of *noise* in how price conveys dividend information. Agents would like to have accurate information about both dividends and supply, so the value of information depends on two information channels that are at odds with each other. This tension is absent from static financial markets because what makes supply valuable information is the iteration of trading.

If the model included an intermediate dividend at $t = 2$, the return between $t = 1$ and $t = 2$

would include a dividend component and the capital gain as above. This would enhance the role of dividend information in determining returns, but supply would still play a part. I solve a general model with intermediate dividends and interim consumption in section 6. I show that the supply information channel remains important even with these additions. For clarity of exposition I present the model with dividends and consumption only at the liquidating stage.

3.2 Portfolio Choice

First I fix the fraction $\lambda \in [0,1]$ of informed agents. Each agent in group j, $j = i, u$ is going to select first-period demand x_1^j and second-period demand x_2^j to maximize first-period expected utility. Doing so he will enjoy value $J^j(W_0, S^j)$ at time one,

$$J^j(W_0, S^j) = \max_{x_1^j, x_2^j} \mathbb{E}\left[-e^{-\delta c_3^j} \middle| \mathcal{F}_1^j\right].$$

where $S^u = P_1$ is the state of the uninformed and $S^i = [P_1, \tilde{\mu}]^T$ is the state vector of the informed. A discount factor does not appear above because there is only one period of consumption. The intertemporal budget constraints give that for $j = i, u$ the final-date consumption is

$$c_3^j = W_0 + x_2^j(D_3 - P_2) + x_1^j(P_2 - P_1).$$

Each agent is going to choose their demand in each period conditional on their information in that period. The decision problem of each agent group is

$$\begin{aligned} J^j(W_0, S^j) &= \max_{x_1^j, x_2^j} \left\{ -e^{-\delta W_0 + \delta x_1^j P_1} \mathbb{E}\left[e^{-\delta x_1^j P_2 + \delta x_2^j P_2} \mathbb{E}\left[e^{-\delta x_2^j D_3} \middle| \mathcal{F}_2^j \right] \middle| \mathcal{F}_1^j \right] \right\} \\ &= -e^{-\delta W_0} \min_{x_1^j} \left\{ e^{\delta x_1^j P_1} \mathbb{E}\left[e^{-\delta x_1^j P_2} \min_{x_2^j} \left\{ e^{\delta x_2^j P_2} \mathbb{E}\left[e^{-\delta x_2^j D_3} \middle| \mathcal{F}_2^j \right] \right\} \middle| \mathcal{F}_1^j \right] \right\}. \end{aligned}$$

The optimal agent demands x_t^{i*} and x_t^{u*} are given in the following proposition.

Proposition 1 (Agent Demand)**.** *For $j = i, u$,*

(i) The agents' second-period demand is

$$x_2^{j*} = \frac{\mathbb{E}\left[D_3 - P_2 | \mathcal{F}_2^j\right]}{\delta \, Var\left(D_3 - P_2 | \mathcal{F}_2^j\right)}$$

(ii) The agents' first-period demand is

$$x_1^{j*} = \frac{\mathbb{E}\left[P_2 - P_1 - h^j(D_3 - P_2) | \mathcal{F}_1^j\right]}{\delta \, Var\left(P_2 - P_1 - h^j(D_3 - P_2) | \mathcal{F}_1^j\right)}$$

where

$$h^j = \frac{Cov\left(D_3 - P_2, P_2 - P_1 \middle| \mathcal{F}_1^j\right)}{Var(D_3 - P_2|\mathcal{F}_1^j)}.$$

In the second period the agents set their demand according to the slope of the return $D_3 - P_2$ in a conditional mean-variance graph. In the first period the agents set their demand similarly. The only difference is that they face risky returns from two periods, $P_2 - P_1$ and $D_3 - P_2$. Each agent hedges in an intertemporal fashion, and this investment behavior is measured by the coefficient h^j. As is standard in the portfolio literature I can separate first-period demand into myopic and hedging components.

Corollary 1 (Decomposition of First-Period Demand)**.** *The agents' first-period demand is*

$$x_1^{j*} = \frac{\mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right]}{\delta\, Var\left(P_2 - P_1|\mathcal{F}_1^j\right)} - \beta^j \frac{\mathbb{E}\left[D_3 - P_2 - \beta^j(P_2 - P_1)|\mathcal{F}_1^j\right]}{\delta\, Var\left(D_3 - P_2 - \beta^j(P_2 - P_1)|\mathcal{F}_1^j\right)}$$

where

$$\beta^j = \frac{Cov\left(D_3 - P_2, P_2 - P_1 \middle| \mathcal{F}_1^j\right)}{Var(P_2 - P_1|\mathcal{F}_1^j)}.$$

The first component is analogous to the second-period demand, where the return is now the myopic return between $t = 1$ and $t = 2$. The second component is the hedging demand of each agent, made up of two terms. The first term is the hedging coefficient of agent group j, β^j. The more each agent thinks distant returns co-vary with immediate returns the more they hedge. Drawing from literature that documents significant negative autocorrelation in returns, I expect that β^j is a negative number; see Fama and French (1988), Lo and MacKinlay (1988) and Poterba and Summers (1988). I revisit β^j in the results section where I show that in equilibrium it is negative.

The second term in the hedging demand involves the residual in the following predictive regression. Suppose that at $t = 2$ each agent group tried to predict the distant returns $D_3 - P_2$ using the more immediate returns $P_2 - P_1$, and at $t = 1$ took into account that they would be running that regression at $t = 2$. In other words, even though $P_2 - P_1$ and $D_3 - P_2$ are not observable at $t = 1$ each agent understands that the world has the structure

$$D_3 - P_2 = \beta^j(P_2 - P_1) + e_3^j \tag{3}$$

where

$$\text{Cov}\left(P_2 - P_1, e_3^j|\mathcal{F}_1^j\right) = 0$$

and tries to exhume as much benefit as his information affords him at $t = 1$. Another interpretation is that every agent forms their demand at $t = 1$ knowing that at $t = 2$ they will be seeing some more information. Therefore at $t = 1$ they incorporate this future arrival of news into their current demand. At $t = 2$ every agent has already seen P_1 so the price P_2 contains the

same news as $P_2 - P_1$. Thus regressing $D_3 - P_2$ onto $P_2 - P_1$ but only on information available at $t = 1$ accounts for the expected arrival of news.

3.3 Market Clearing

At each point in time the total agent demand must equal total supply,

$$\lambda x_t^{i*} + (1-\lambda) x_t^{u*} = \tilde{\theta}_t.$$

Imposing market clearing at each point in time, substituting for agent demand and rearranging for prices gives an expression that I compare to the price conjecture. By matching coefficients I obtain expressions for the information sensitivities $p_{\mu\theta}$ and $q_{\mu\theta}$.

Proposition 2 (Equilibrium Price Coefficient Ratios)**.** *In equilibrium, for a fixed fraction of informed investors λ,*

(i) The second-period information sensitivity $p_{\mu\theta}$ is

$$p_{\mu\theta} = -\frac{\lambda}{\delta \sigma_\zeta^2}. \tag{4}$$

(ii) The first-period information sensitivity $q_{\mu\theta}$ is the solution to the equation

$$\begin{aligned} &\left[\delta p_{\mu\theta}^2 \left(p_{\mu\theta} - \rho q_{\mu\theta}\right) + \delta \frac{\sigma_\eta^2}{\sigma_\zeta^2}(p_{\mu\theta} - q_{\mu\theta}) - \left(\frac{1}{\sigma_\zeta^2} + \delta p_{\mu\theta}\right)\left(p_{\mu\theta} - \rho q_{\mu\theta}\right)\left(p_{\mu\theta} - q_{\mu\theta}\right)\right] \\ &= \left[p_{\mu\theta}\left(p_{\mu\theta} - \rho q_{\mu\theta}\right) - \delta \sigma_\eta^2 \left(p_{\mu\theta} - q_{\mu\theta}\right)\right]\left[\frac{\left(p_{\mu\theta} - \rho q_{\mu\theta}\right)^2}{\sigma_\eta^2} + \frac{q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2}{\sigma_\mu^2 \sigma_\theta^2}\right] \end{aligned} \tag{5}$$

The financial market equilibrium exists when equation (5) has a real root. Moreover, it is unique when this real root is unique. Because (5) is a cubic equation in $q_{\mu\theta}$, existence is always guaranteed. Uniqueness is assured by the condition that the discriminant of (5), $\Delta^{q_{\mu\theta}}$, is non-positive, because in that case it is well-known that (5) has at most one real root. In addition, because q_μ is the coefficient of mean dividend in prices I require that $q_\mu \geq 0$ and because q_θ is the coefficient of supply in prices I require that $q_\theta < 0$. This imposes the additional requirement that $q_{\mu\theta}$ is not positive. The next proposition summarizes results on existence and uniqueness.

Proposition 3 (Existence and Uniqueness)**.** *For a fixed fraction of informed investors λ,*

(i) A financial market equilibrium at λ always exists.

(ii) There always exists a financial market equilibrium at λ such that

(a) if $\lambda = 0$, then $q_{\mu\theta} = 0$,

(b) if $\lambda > 0$, then $q_{\mu\theta} < 0$.

(iii) The financial market equilibrium at λ is unique if the discriminant

$$\Delta^{q_{\mu\theta}} = 18\gamma_3\gamma_2\gamma_1\gamma_0 - 4\gamma_2^3\gamma_0 + \gamma_2^2\gamma_1^2 - 4\gamma_3\gamma_1^3 - 27\gamma_3^2\gamma_0^2$$

is non-positive, where γ_3, γ_2, γ_1 and γ_0 are the coefficients of the powers of $q_{\mu\theta}$ in the polynomial version of (5),

$$\gamma_3 q_{\mu\theta}^3 + \gamma_2 q_{\mu\theta}^2 + \gamma_1 q_{\mu\theta} + \gamma_0 = 0. \tag{6}$$

The coefficient γ_i, $i = 0, \ldots 3$, is a polynomial of order $4 - i$ in $p_{\mu\theta}$. I give γ_3, γ_2, γ_1 and γ_0 explicitly in terms of the model parameters and $p_{\mu\theta}$ in the appendix.

For every solution, example and graph that I provide in this paper I have checked through the $\Delta^{q_{\mu\theta}}$ non-positivity condition that each equilibrium is unique and that $q_\mu \geq 0$ and $q_\theta < 0$.[7] Finally, there is an important special case in which I can solve for $q_{\mu\theta}$ explicitly. When supply follows a random walk it follows by inspection of (5) that $q_{\mu\theta} = p_{\mu\theta}$.

Corollary 2 (Random-Walk-Supply Solution)**.** *When $\rho = 1$, $p_{\mu\theta} - \rho q_{\mu\theta} = 0$.*

Using the ratios $p_{\mu\theta}$ and $q_{\mu\theta}$ I finish the computation of the equilibrium by solving for the price coefficients. I give expressions for the price coefficients in proposition 9 of the appendix, whereas here I show graphs of them in figures 2 and 3.

The coefficients of mean dividends and inferred mean dividends are positive because higher dividends command higher prices. When $\lambda = 0$, $q_\mu = 0 = p_\mu$ because when there are no informed agents in the economy their information does not determine prices at all. As λ increases the coefficients of $\tilde{\mu}$ increase because the impact of the informed agents in the economy increases as there are more of them. A similar reasoning for the uninformed explains why $q_{\hat{\mu}} = 0 = p_{\hat{\mu}}$ at $\lambda = 1$ and why as λ increases the coefficients of $\hat{\mu}_t$ decrease.

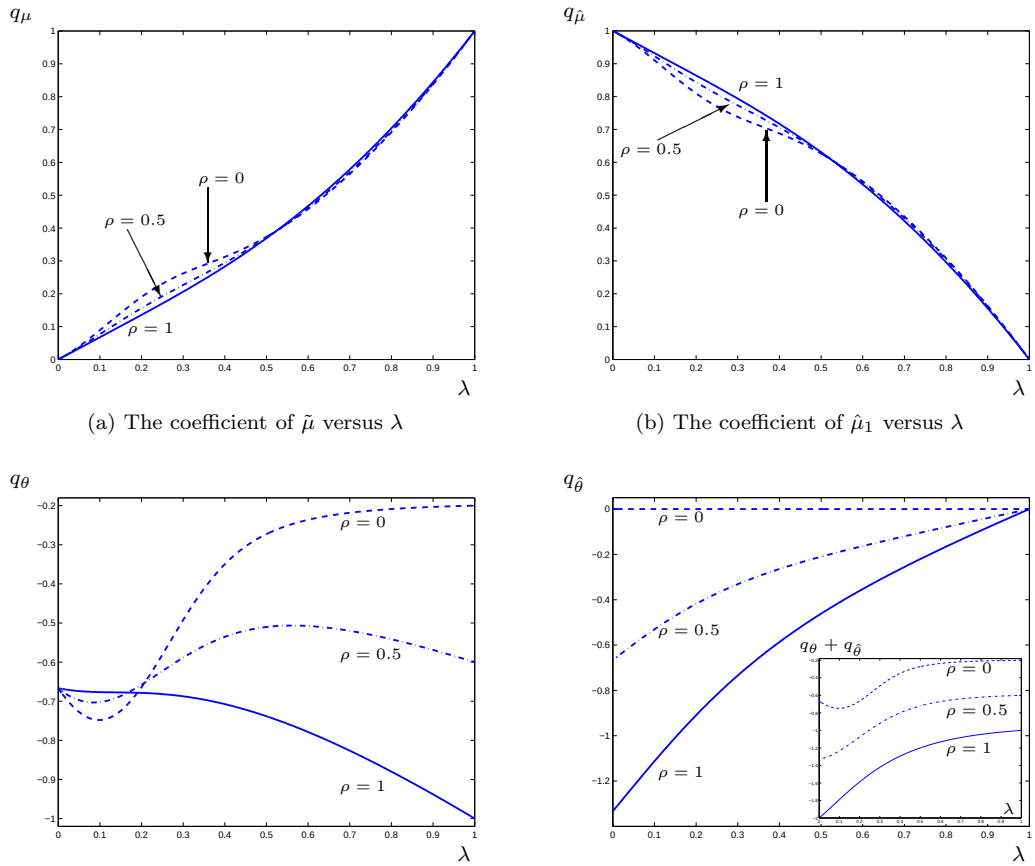
The coefficients of supply are negative because an increase in the level of supply means that prices drop in equilibrium. The second-period coefficient of supply is increasing in λ reflecting that as there are more informed agents prices become more informative about liquidating dividends and therefore the economy is overall less risky. The first-period coefficient of supply seems to have monotonicity that is very sensitive to the value of ρ. Notice, however, that there are also indirect supply effects through the inference of the supply level.[8] Using that $y_1 \in \mathcal{F}_1^u$ and $y_2 \in \mathcal{F}_2^u$ I can write prices in a manner that does not involve the inference over supply,

$$\begin{aligned} P_1 &= \left(q_\mu + q_{\mu\theta} q_{\hat{\theta}}\right) \tilde{\mu} + \left(q_\theta + q_{\hat{\theta}}\right) \tilde{\theta}_1 + \left(q_{\hat{\mu}} - q_{\mu\theta} q_{\hat{\theta}}\right) \hat{\mu}_1, \\ P_2 &= \left(p_\mu + p_{\mu\theta} p_{\hat{\theta}}\right) \tilde{\mu} + \left(p_\theta + p_{\hat{\theta}}\right) \tilde{\theta}_2 + \left(p_{\hat{\mu}} - p_{\mu\theta} p_{\hat{\theta}}\right) \hat{\mu}_2. \end{aligned}$$

In this particular price representation the coefficient of supply in the first period, $q_\theta + q_{\hat{\theta}}$, has non-monotonic patterns in λ only for very low values of ρ (see the inlay in figure 2d). This is

[7] At the time of this writing, I have not found counterexamples to existence, uniqueness, $q_\mu \geq 0$ and $q_\theta < 0$.

[8] In the second period there are no indirect supply effects because $p_{\hat{\theta}} = 0$. See proposition 9 of the appendix.



(a) The coefficient of $\tilde{\mu}$ versus λ

(b) The coefficient of $\hat{\mu}_1$ versus λ

(c) The coefficient of $\tilde{\theta}_1$ versus λ

(d) The coefficient of $\hat{\theta}_1$ versus λ. Inlay: $q_\theta + q_{\hat{\theta}}$ versus λ

Figure 2: Price coefficients at $t = 1$ for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$.

a manifestation of the informativeness trade-off that I describe below. This representation of prices does not change the monotonicity of any other price coefficients.

4 Information Market Equilibrium

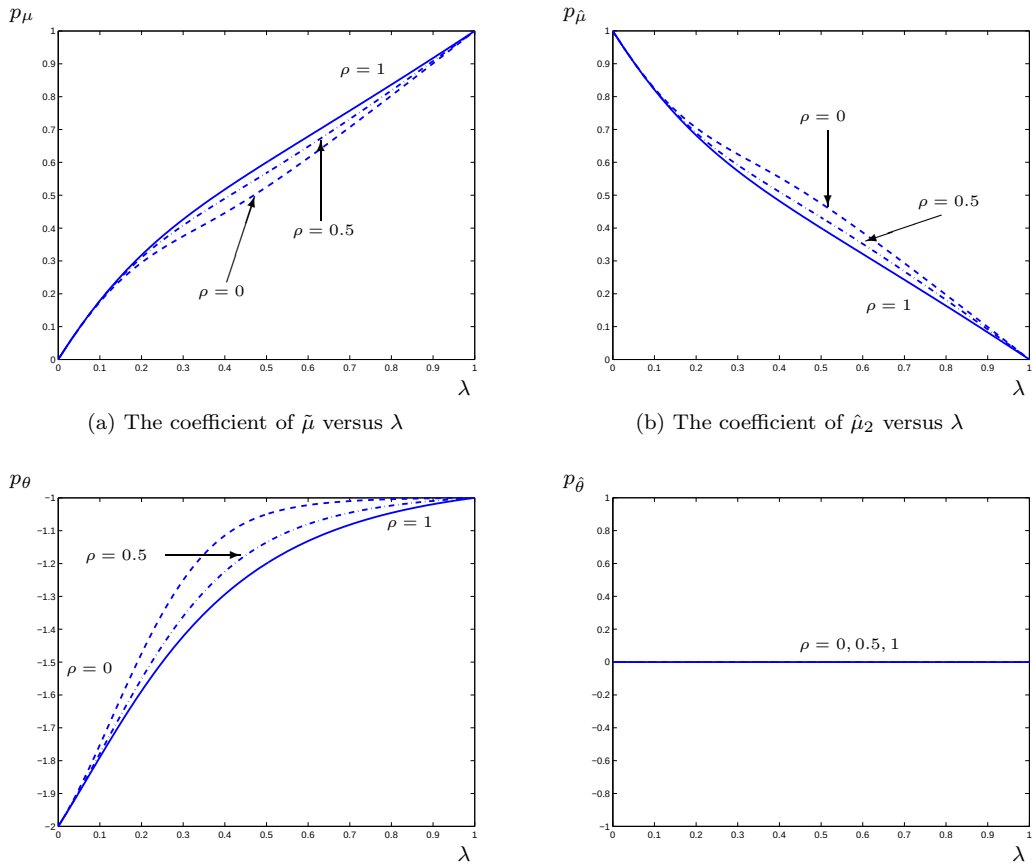
Having obtained the equilibrium in the financial market for each fixed fraction λ, I step one period back to $t = 0$ in order to endogenize λ. A candidate equilibrium in the information market is a tuple of an exogenous cost of information κ_0, a fraction of informed investors $\lambda \in [0, 1]$ and a Financial Market Equilibrium $\left(P_1^\lambda, P_2^\lambda\right)$.

Definition 2 (Information Market Equilibrium)**.**

(a) If $\lambda^* \in [0, 1]$ and

$$\mathbb{E}\left[J^u(W_0, P_1^{\lambda^*}; \lambda^*)\middle|\mathcal{F}_0^u\right] = \mathbb{E}\left[J^i(W_0 - \kappa_0, P_1^{\lambda^*}, \tilde{\mu}; \lambda^*)\middle|\mathcal{F}_0^u\right]$$

then $\left(\kappa_0, \lambda^*, \left(P_1^{\lambda^*}, P_2^{\lambda^*}\right)\right)$ is an Information Market Equilibrium.



(a) The coefficient of $\tilde{\mu}$ versus λ

(b) The coefficient of $\hat{\mu}_2$ versus λ

(c) The coefficient of $\tilde{\theta}_2$ versus λ

(d) The coefficient of $\hat{\theta}_2$ versus λ

Figure 3: Price coefficients at $t = 2$ for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$. The coefficient of $\hat{\theta}_2$ is always zero.

(b) If $\mathbb{E}\left[J^u(W_0, P_1^0; 0)\middle|\mathcal{F}_0^u\right] > \mathbb{E}\left[J^i(W_0 - \kappa_0, P_1^0, \tilde{\mu}; 0)\middle|\mathcal{F}_0^u\right]$ then $\left(\kappa_0, 0, \left(P_1^0, P_2^0\right)\right)$ is an Information Market Equilibrium.

(c) If $\mathbb{E}\left[J^u(W_0, P_1^1; 1)\middle|\mathcal{F}_0^u\right] < \mathbb{E}\left[J^i(W_0 - \kappa_0, P_1^1, \tilde{\mu}; 1)\middle|\mathcal{F}_0^u\right]$ then $\left(\kappa_0, 1, \left(P_1^1, P_2^1\right)\right)$ is an Information Market Equilibrium.

At $t = 0$, before the agents decide on their information status, everyone is uninformed, has wealth W_0 and information set $\mathcal{F}_0^u$. If an agent decides to remain uninformed he will enjoy value given by the value function J^u. If an agent decides to become informed he will pay κ_0 and he will switch value functions to J^i. But $J^u(W_0, P_1^\lambda; \lambda)$ and $J^i(W_0 - \kappa_0, P_1^\lambda, \tilde{\mu}; \lambda)$ are random variables; before an agent sees P_1^λ or $\tilde{\mu}$ he does not know what their realization will be. Thus the comparison of values at $t = 0$ is done conditional on $\mathcal{F}_0^u$.

Each type of agent anticipates rationally the workings of the financial market and compares the benefit of being informed versus the cost of giving up κ_0. The benefit of being informed depends on the fraction λ of agents that are informed, because how many informed agents exist influences the informativeness of prices about $\tilde{\mu}$, $\tilde{\theta}_1$ and $\tilde{\theta}_2$. To facilitate the derivation of this equilibrium, I define the value of information at λ as the relative value of being informed

against being uninformed.

Definition 3 (Value of Information)**.** The *value of information* $\psi_0(\lambda)$ is the relative certainty-equivalent value of being informed,

$$\mathbb{E}\left[J^u\left(W_0, P_1^\lambda, \tilde{\mu}; \lambda\right)\Big|\mathcal{F}_0^u\right] = \mathbb{E}\left[J^i\left(W_0 - \psi_0(\lambda), P_1^\lambda; \lambda\right)\Big|\mathcal{F}_0^u\right].$$

Due to CARA utility I can write

$$e^{\delta\psi_0(\lambda)} = \frac{\mathbb{E}\left[J^u(W_0, P_1^\lambda, \tilde{\mu}; \lambda)\Big|\mathcal{F}_0^u\right]}{\mathbb{E}\left[J^i(W_0, P_1^\lambda; \lambda)\Big|\mathcal{F}_0^u\right]}$$

and thus in order to calculate $\psi_0(\lambda)$ I need only calculate the conditional expectations of each value function conditional on prior information. The CARA-normal environment allows this calculation in closed form in moments of returns. I provide the details in the appendix.

The cost of information acquisition, κ_0, is an exogenous parameter. The equilibrium fraction of informed agents λ^* is such that every agent finds that the value of information is the same as its cost,

$$\kappa_0 = \psi_0(\lambda^*).$$

If $\kappa_0 > \psi_0(0)$ then $\lambda^* = 0$ is an equilibrium and if $\kappa_0 < \psi_0(1)$ then $\lambda^* = 1$ is an equilibrium. Thus in order to determine λ^* I need to derive the entire value-of-information curve as a function of λ. I give this curve in the main theorem of the paper, where I drop dependence on λ for notational clarity.

Theorem 1 (Value of Information)**.** *The value of information is*

$$e^{2\delta\psi_0} = \frac{Var(D_3 - P_2|\mathcal{F}_2^u)}{Var(D_3 - P_2|\mathcal{F}_2^i)}\frac{Var\left(P_2 - P_1 - h^u(D_3 - P_2)|\mathcal{F}_1^u\right)}{Var\left(P_2 - P_1 - h^i(D_3 - P_2)|\mathcal{F}_1^i\right)}.$$

There are two terms in the value of information, each coming from a different trading period in the model. The first term above is the value of information in a Grossman and Stiglitz (1980) economy starting from $t = 2$. The better the uninformed agents know the dividends compared to the informed, the smaller the value of information. The second term is a multiplicative correction on top of the Grossman-Stiglitz term. It is analogous to the first term because each term that appears inside the conditional variances is the information that agents use to form their demand. The better the uninformed know this information in relation to the informed, the lower the value of information.

There is one caveat in this interpretation. The Grossman-Stiglitz fraction does not reflect entirely all the information effects at $t = 2$ because it measures information as if the economy started at $t = 2$. At that time, however, the agents have already seen some information at $t = 1$. Moreover, at $t = 1$ they know that at $t = 2$ they will be accounting for information already seen. Therefore at $t = 1$ they try to predict how at $t = 2$ they will be accounting for

information already seen. To write the value of information in a way that reflects these effects, I use the decomposition of first-period agent demand into myopic and hedging parts.

Corollary 3. *The value of information can be written as*

$$e^{2\delta\psi_0} = \frac{\frac{Var(D_3-P_2|\mathcal{F}_2^u)}{Var(D_3-P_2|\mathcal{F}_1^u)}}{\frac{Var(D_3-P_2|\mathcal{F}_2^i)}{Var(D_3-P_2|\mathcal{F}_1^i)}} \frac{Var\left(D_3-P_2-\beta^u(P_2-P_1)|\mathcal{F}_1^u\right)}{Var\left(D_3-P_2-\beta^i(P_2-P_1)|\mathcal{F}_1^i\right)} \frac{Var\left(P_2-P_1|\mathcal{F}_1^u\right)}{Var\left(P_2-P_1|\mathcal{F}_1^i\right)}. \tag{7}$$

The value of information is

$$\begin{aligned}\psi_0 = &\frac{1}{2\delta}\log\frac{\mathrm{Var}(D_3-P_2|\mathcal{F}_2^u)}{\mathrm{Var}(D_3-P_2|\mathcal{F}_2^i)} - \frac{1}{2\delta}\log\frac{\mathrm{Var}(D_3-P_2|\mathcal{F}_1^u)}{\mathrm{Var}(D_3-P_2|\mathcal{F}_1^i)}\\ &+\frac{1}{2\delta}\log\frac{\mathrm{Var}\left(D_3-P_2-\beta^u(P_2-P_1)|\mathcal{F}_1^u\right)}{\mathrm{Var}\left(D_3-P_2-\beta^i(P_2-P_1)|\mathcal{F}_1^i\right)}\\ &+\frac{1}{2\delta}\log\frac{\mathrm{Var}\left(P_2-P_1|\mathcal{F}_1^u\right)}{\mathrm{Var}\left(P_2-P_1|\mathcal{F}_1^i\right)}.\end{aligned}$$

The first two terms measure the value of information coming from second-period returns. In the first period agents obtain some information about second-period returns from prices. They are only willing to pay for information they obtain at $t = 2$ that increments what they already know. Thus the intertemporal difference of relative variances accounts for past price information. In addition, the agents anticipate at $t = 1$ that at $t = 2$ they will be accounting for past information. The value of information coming from how well they can anticipate this is measured by the third term. If by becoming informed the agents can predict better the future accounting of past information, they are willing to pay more for information. The last term is the relative variance of the first-period myopic return. This is the channel where the supply information effect shows up. Supply fluctuations $\tilde{\theta}_2 - \tilde{\theta}_1$ determine $P_2 - P_1$, so when $\tilde{\theta}_2 - \tilde{\theta}_1$ is predictable by $\tilde{\theta}_1$ how well agents know $\tilde{\theta}_1$ matters for the value of information. Namely, if by becoming informed the agents gain in how well they estimate $\tilde{\theta}_1$, they are willing to pay more for information.

Finally, I note that if the cost of acquiring information is arbitrarily small, it is always better to be informed.

Proposition 4 (The Value of Information Is Positive)**.** *For* $\lambda \in [0, 1]$, $\psi_0(\lambda) > 0$.

5 Results

5.1 The Informativeness Trade-Off

The dynamic nature of the financial market enters the value of information through the informativeness of first-period prices P_1 about returns $D_3 - P_2$ and $P_2 - P_1$. These returns generally load on both dividends and supply, so prices contain information about them through two distinct channels. I claim that as the number of informed investors in the market increases, the

uninformed investors' estimation of the dividend improves but their estimation of the supply worsens. Proposition 5 establishes that inference qualities, and thus price informativenesses, move in opposite directions.

Proposition 5 (Informativeness Trade-Off)**.** *Holding σ_μ^2 and σ_θ^2 fixed, Var$\big(\tilde{\theta}_1|\mathcal{F}_1^u\big)$ is decreasing in Var$(\tilde{\mu}|\mathcal{F}_1^u)$.*

The informativeness trade-off says that if I vary *any parameter* of the model other than σ_μ^2 and σ_θ^2 in a way that makes Var $(\tilde{\mu}|\mathcal{F}_1^u)$ decrease then Var$\big(\tilde{\theta}_1|\mathcal{F}_1^u\big)$ will increase and vice versa.[9] For the purposes of the information equilibrium that I have just described the parameter that I vary is λ. I prove this proposition formally in the appendix whereas here I give an intuitive explanation. Consider the price signal at $t = 1$, $y_1 = q_\mu\tilde{\mu} + q_\theta\tilde{\theta}_1$. It is a noisy signal of $\tilde{\mu}$ because of the presence of $\tilde{\theta}_1$. It is important to note, however, that the magnitude of noise in this signal depends on the coefficients q_μ and q_θ. Moreover, q_μ and q_θ are functions of the number of informed investors λ and thus how much noise prices contain depends on λ.

Recall that the information sensitivity $q_{\mu\theta}$ measures how much prices respond to changes in dividend information. For higher numbers of informed agents prices respond more to changes in dividend information, because there is more dividend information in the market. In other words, the magnitude of the ratio $q_{\mu\theta}$ is increasing in λ.

In equilibrium q_μ and q_θ are known by every investor. Therefore when the uninformed see y_1 they can divide it by q_μ and thus glean the quantity

$$\tilde{\mu} + \frac{1}{q_{\mu\theta}}\tilde{\theta}_1.$$

If the number of informed investors increases the magnitude of $1/q_{\mu\theta}$ decreases. Then prices become a signal of $\tilde{\mu}$ with higher precision, because the total noise in price, $\tilde{\theta}_1/q_{\mu\theta}$, has lower variance.

At the same time one can think of prices as a noisy signal of stochastic supply. From this perspective $\tilde{\mu}$ is noise with respect to $\tilde{\theta}_1$. When the uninformed see y_1 they can also divide it by q_θ and thus observe the quantity

$$q_{\mu\theta}\tilde{\mu} + \tilde{\theta}_1.$$

Here, if the number of informed investors increases so does the magnitude of $q_{\mu\theta}$. Therefore prices become a signal of $\tilde{\theta}_1$ with lower precision, because the total noise in price, $q_{\mu\theta}\tilde{\mu}$, has *higher* variance.

An alternative way to think about the information trade-off is to look at the amount of information in prices as a weighted average of dividend information and supply information. As above, dividing y_1 by any constant known in equilibrium does not change the amounts of

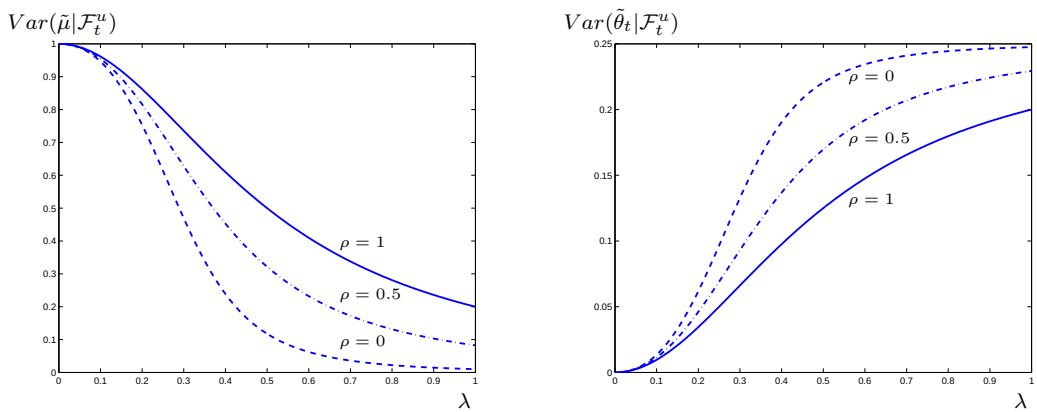
[9] The information trade-off can also be stated in terms of sensitivity of price information. The sensitivity of price information to supply information is $(q_{\mu\theta})^{-1}$, which is the inverse of the sensitivity of price information to dividend information. Therefore as prices become more sensitive to dividend information they become less sensitive to supply information.

information that it conveys. Dividing y_1 by its standard deviation gives

$$\frac{y_1}{\sqrt{\mathrm{Var}(y_1)}} = \frac{q_\mu \tilde{\mu} + q_\theta \tilde{\theta}_1}{\sqrt{q_\mu^2 \sigma_\mu^2 + q_\theta^2 \sigma_\theta^2}} = \frac{q_{\mu\theta}\sigma_\mu}{\sqrt{q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2}} \frac{\tilde{\mu}}{\sigma_\mu} + \frac{\sigma_\theta}{\sqrt{q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2}} \frac{\tilde{\theta}_1}{\sigma_\theta}.$$

The standardized information content of prices is a linear combination of two independent standard normal random variables. Notice that the squares of the coefficients of each standard normal add up to one. Therefore, each coefficient measures how much weight the amount of each type of information carries in the overall amount of information contained in prices. This shows that changes in the economy that increase the amount of dividend information contained in prices decrease the amount of supply information contained in prices and vice versa.

I plot the inference qualities of mean dividends and supply in figure 4. As the number of informed investors in the market increases, prices become more informative about mean dividends and thus $\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)$ decreases. At the same time $\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right)$ increases, which means that prices become less informative about supply. Figure 4 also shows that the informativeness of price about dividends and supply shift in response to changes in the persistence of supply. For a fixed number of informed investors λ, when ρ decreases the uninformed can predict changes in supply better. The uninformed then have an informational advantage with respect to supply, so $\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right)$ must adjust upwards in order to support λ in equilibrium. The information trade-off then implies that $\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)$ decreases.



(a) How well $\tilde{\mu}$ is known at $t = 1$

(b) How well $\tilde{\theta}_1$ is known at $t = 1$

Figure 4: Informativeness of price for dividends and supply at $t = 1, 2$ for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$.

5.2 Complementarities

Here I highlight why adding another trading period to the economy of Grossman and Stiglitz (1980) gives complementarities in information acquisition. Complementarities obtain when the slope of the value of information $\psi_0(\lambda)$ with respect to the number of informed traders λ is positive. That (5) is a cubic polynomial in λ presents a bottleneck of tractability in

$\partial\psi_0(\lambda)/\partial\lambda$ for arbitrary λ. Nevertheless the value of information is in closed form in moments of $D_3 - P_2$ and $P_2 - P_1$, even though these moments are not themselves a closed form of λ. First I explain the result directly in terms of $D_3 - P_2$ and $P_2 - P_1$. In section 5.2.1 I prove the existence of information complementarities when supply is independent over time and I give an approximation of $\partial\psi_0(\lambda)/\partial\lambda$ for small λ and any ρ.

The information trade-off in $\tilde{\mu}$ and $\tilde{\theta}_1$ creates a similar information trade-off in cash flows and capital gains. The second-period return is a *liquidating capital gain*,

$$D_3 - P_2 = \tilde{\zeta} - p_\eta \tilde{\eta} + p_\mu^D (\tilde{\mu} - \hat{\mu}_1) + p_\theta^D \tilde{\theta}_1 \tag{8}$$

where I define

$$\begin{aligned} p_\theta^D &= -\rho p_\theta, \\ p_\mu^D &= p_{\hat{\mu}} \frac{\mathrm{Var}(\tilde{\mu}|\mathcal{F}_2^u)}{\mathrm{Var}(\tilde{\mu}|\mathcal{F}_1^u)}, \\ p_\eta &= p_\theta + p_{\hat{\mu}}(p_{\mu\theta} - \rho q_{\mu\theta}) \frac{\mathrm{Var}(\tilde{\mu}|\mathcal{F}_2^u)}{\sigma_\eta^2}. \end{aligned}$$

The first term and the second term on the right-hand side of (8) depend on $\tilde{\zeta}$ and $\tilde{\eta}$, against which both agents groups are symmetrically uninformed. Therefore any contribution to the value of information coming from these terms also measures exposure to risk. Higher risk means that information is more valuable, but at the same time higher risk means that agents will hold less of the asset. There is also dependence on $\hat{\mu}_1$, but this is common information. Therefore any relative value of information coming purely from information asymmetry is going to come through the variables $\tilde{\mu}$ and $\tilde{\theta}_1$. How much the supply information matters depends on the value of ρ. When supply is independent over time the price informativeness of supply does not matter for $D_3 - P_2$.

The first-period myopic return is the *capital gain from trade*,

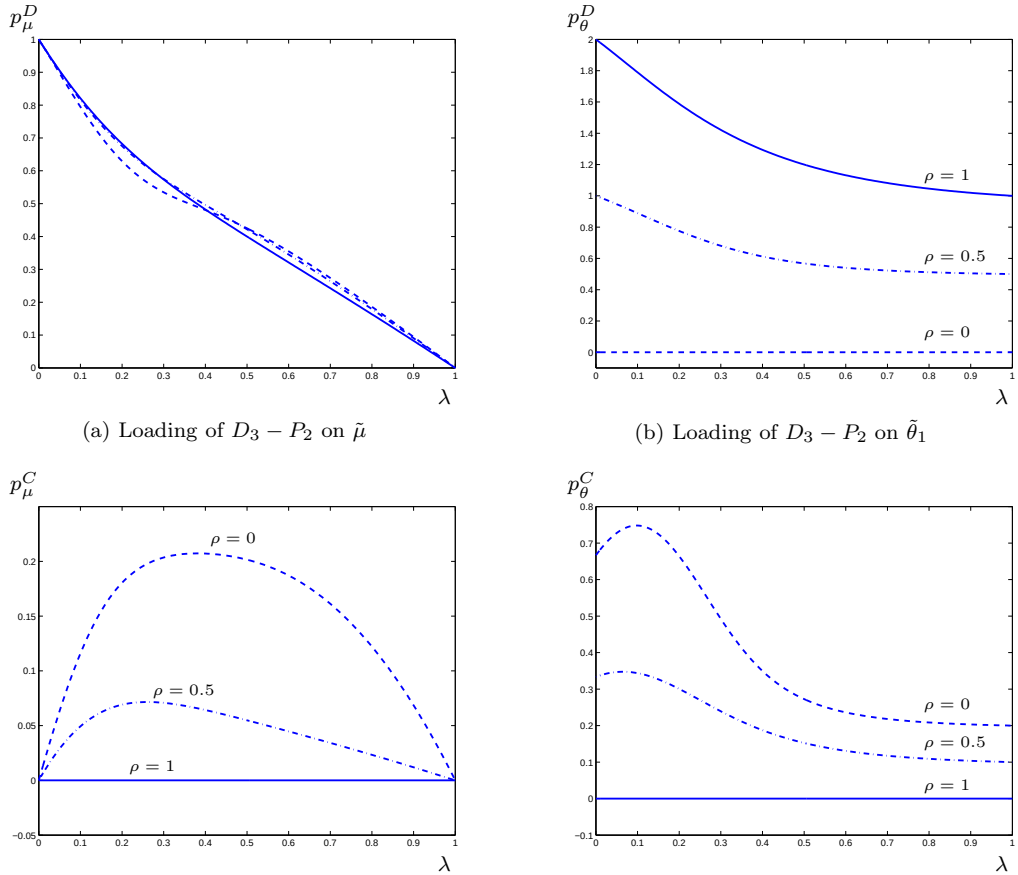
$$P_2 - P_1 = p_\eta \tilde{\eta} + p_\mu^C (\tilde{\mu} - \hat{\mu}_1) + p_\theta^C \tilde{\theta}_1 \tag{9}$$

where I define

$$\begin{aligned} p_\theta^C &= \rho p_\theta - (q_\theta + q_{\hat{\theta}}), \\ p_\mu^C &= (p_{\mu\theta} - \rho q_{\mu\theta}) p_\eta + q_{\mu\theta} p_\theta^C. \end{aligned}$$

Similarly to above, the relative value of information purely due to information asymmetry about $P_2 - P_1$ comes through $\tilde{\mu}$ and $\tilde{\theta}_1$. Again ρ regulates how much but here the effect has a different direction in ρ. When supply is independent over time the price informativeness of supply matters strongly, because then the coefficient of $\tilde{\theta}_1$ is exactly q_θ (see proposition 9).

I show the loadings of liquidating capital gains and capital gains from trade on the two



(a) Loading of $D_3 - P_2$ on $\tilde{\mu}$

(b) Loading of $D_3 - P_2$ on $\tilde{\theta}_1$

(c) Loading of $P_2 - P_1$ on $\tilde{\mu}$

(d) Loading of $P_2 - P_1$ on $\tilde{\theta}_1$

Figure 5: Return coefficients for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$.

unknowns in figure 5. In the case of the random walk assumption for supply, $\rho = 1$, capital gains from trade do not load at all on supply but liquidating capital gains do.[10] As ρ decreases supply determines liquidating capital gains less but it determines capital gains from trade more. When supply is independent over time, $\rho = 0$, capital gains from trade load positively on supply but liquidating capital gains do not at all. Because capital gains from trade are increasingly identified with $\tilde{\theta}_1$ as ρ decreases, any information effects that are more dominant as ρ decreases are coming from capital gains from trade, not liquidating capital gains.

Furthermore, at $t = 1$ each agent is mostly concerned with capital gains from trade. This can be seen from the demand equations, where the myopic part is based only on $P_2 - P_1$. $D_3 - P_2$ plays a secondary role for two reasons. Firstly, each agent hedges it out against $P_2 - P_1$. Secondly, whatever agents do to manage the risk in $D_3 - P_2$ at $t = 1$ they get to correct after they have some more information at $t = 2$.

In the previous section I established that as λ increases the value of knowing dividend information decreases but the value of knowing supply information increases. The former effect

[10] Note that $p_\mu^C = 0$ when $\rho = 1$. I explain this pattern and its importance in section 5 below.

is the standard result of Grossman and Stiglitz (1980). Pushing against this effect is a supply information effect that works through $P_2 - P_1$. As ρ decreases capital gains from trade are more predictable by $\tilde{\theta}_1$ because they load more on it. Therefore as ρ decreases the value of information coming from capital gains from trade becomes an increasing function of λ. The two effects combine in a way that gives a value of information that is nonmonotonic in λ.

I exhibit this pattern in figure 6, where I plot the value of information for two extreme values of ρ and one intermediate case: supply is a random walk, $\rho = 1$, supply is mean-reverting, $\rho = 0.5$, and supply is independent of its past, $\rho = 0$. The remaining parameters of the model are $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. When $\rho = 1$ the value of information is decreasing in λ because the dividend information effect dominates. As ρ decreases the supply information effect starts to become more pronounced. The combination of the two effects gives a value of information that is increasing for low λ and decreasing for larger λ. For low ρ, when few informed investors exist in the market returns are mostly determined by noisy stochastic supply. As more informed investors enter the economy, they start making prices less informative about supply, pushing the value of information upwards but more informative about dividends, pushing the value of information downwards. Eventually when λ is large enough the dividend information effect dominates, twisting the complementarity of low λ into a substitutability for high λ. Moreover, because the importance of the supply effect in the value of information increases as ρ decreases, the range of dominance of the supply effect keeps increasing as ρ decreases. As figure 6 shows, when $\rho = 0.5$, $\psi_0(\lambda)$ is increasing for $\lambda \leq 0.06$, but when $\rho = 0$ the value of information is increasing for $\lambda \leq 0.105$, namely for a larger range of λ.

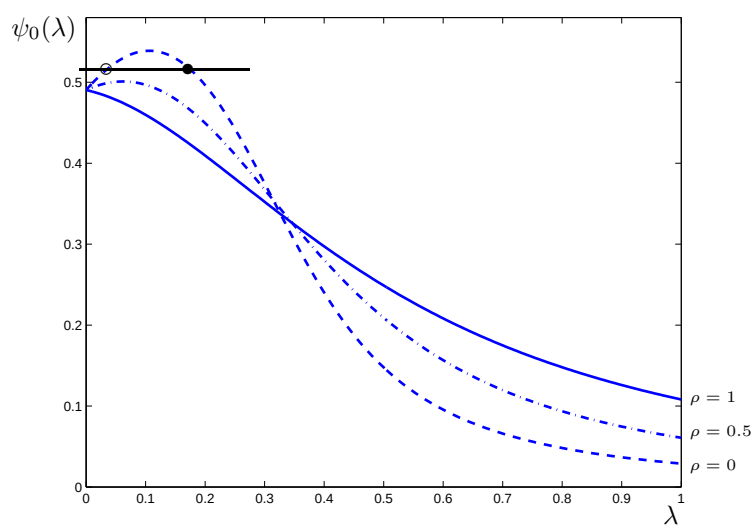


Figure 6: The value of information for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$. For $\rho = 0$, when $\kappa_0 = 0.52$ there is one interior equilibrium at $\lambda_* = 0.04$ (empty circle) and another interior equilibrium at $\lambda^* = 0.17$ (filled circle).

5.2.1 The Slope of the Value of Information

In this section I prove that when supply is independent over time complementarities in information acquisition are always present. Moreover, I develop an approximation of the slope of the value of information in the number of informed investors, $\partial\psi_0(\lambda)/\partial\lambda$, for arbitrary supply persistence ρ.

Theorem 2 (Information Complementarities)**.** *When supply is independent over time, the value of information is increasing for small numbers of informed agents,*

$$\left.\frac{\partial e^{2\delta\psi_0(\lambda)}}{\partial\lambda}\right|_{\lambda=0} = 2\frac{\sigma_\mu^2}{\sigma_\zeta^4}\frac{\frac{1}{\sigma_\zeta^2}+\delta^2\sigma_\eta^2}{\left(\frac{1}{\sigma_\zeta^2+\sigma_\mu^2}+\delta^2\sigma_\eta^2\right)^2} > 0.$$

This theorem gives the sign of the slope of the value of information at the origin because $e^{2\delta\psi_0(\lambda)}$ is a positive monotone transformation of $\psi_0(\lambda)$. The intuition is that as informed agents enter the economy, agents who decide to remain uninformed lose accuracy of supply information. In addition, although the presence of more informed agents improves the accuracy of public dividend information, it also implies that whoever remains uninformed must face more traders with superior information. Overall, uninformed agents lose more from diminished predictability of supply fluctuations than what they gain from improved dividend information. Therefore, as more agents become informed the remaining uninformed agents are willing to pay more for information.

In order to approximate $\partial\psi_0(\lambda)/\partial\lambda$ for arbitrary ρ, I approximate the solution to polynomial (6) by ignoring terms of order λ^2 and higher. Doing so replaces (6) with a linear equation in λ, which I solve for the first-period information sensitivity $q_{\mu\theta}$. As the appendix shows, this approximation is good around $\lambda = 0$ and improves as ρ decreases from one. Using this approximate solution for $q_{\mu\theta}$, I carry out all the subsequent steps in the calculation of the value of information as if $q_{\mu\theta}$ was exact. This gives an approximation to the slope of the value of information which is very good at $\lambda = 0$.

Proposition 6 (Approximate Slope of the Value of Information)**.** *Within a linear approximation of the first-period information sensitivity $q_{\mu\theta}$ in λ,*

$$\left.\frac{\partial e^{2\delta\psi_0(\lambda)}}{\partial\lambda}\right|_{\lambda=0} \approx \frac{2}{\sigma_\zeta^4}\frac{1}{\frac{1}{\sigma_\zeta^2+\sigma_\mu^2}+\delta^2\sigma_\eta^2}\left[\frac{\frac{1}{\sigma_\zeta^2}+\delta^2\sigma_\eta^2}{\frac{1}{\sigma_\zeta^2+\sigma_\mu^2}+\delta^2\sigma_\eta^2}\sigma_\mu^2 - \rho\frac{\sigma_\zeta^2+\sigma_\mu^2}{\sigma_\zeta^2}\sigma_\theta^2\right].$$

The smaller ρ is, the better the approximation.

Here the adverse effect of increasing ρ on the presence of complementarities is clear. As supply becomes more persistent, ρ increases and the complementarity in the value of information for small numbers of informed agents becomes less pronounced. When supply is independent, $\rho = 0$, the above approximation is exact.

5.2.2 Comparative Statics on the Value of Information

Here I explore how the value of information changes in response to changes in ex-ante uncertainty parameters. Figure 7a shows comparative statics with respect to uncertainty in dividend information. The entire curve of the value of information shifts upward when σ_μ increases because information is more valuable when there is higher uncertainty about dividends. But because prices become more informative about dividends as λ increases, the increase in the value of information due to higher σ_μ is smaller for higher λ.

The effect of changes in the residual uncertainty σ_ζ is subject to two opposing forces. On the one hand, information is more valuable when residual uncertainty is lower because agents who decide to be informed have access to more accurate dividend information. On the other hand, lower σ_ζ also makes information less valuable because the informed transmit more accurate information through prices.[11] The former effect dominates when few informed agents exist because the aggregate amount of information that uninformed agents receive is still small. But as figure 7b shows, the latter effect becomes more important as the number of informed agents increases, making the value of information non-monotonic in σ_ζ for high numbers of informed agents.

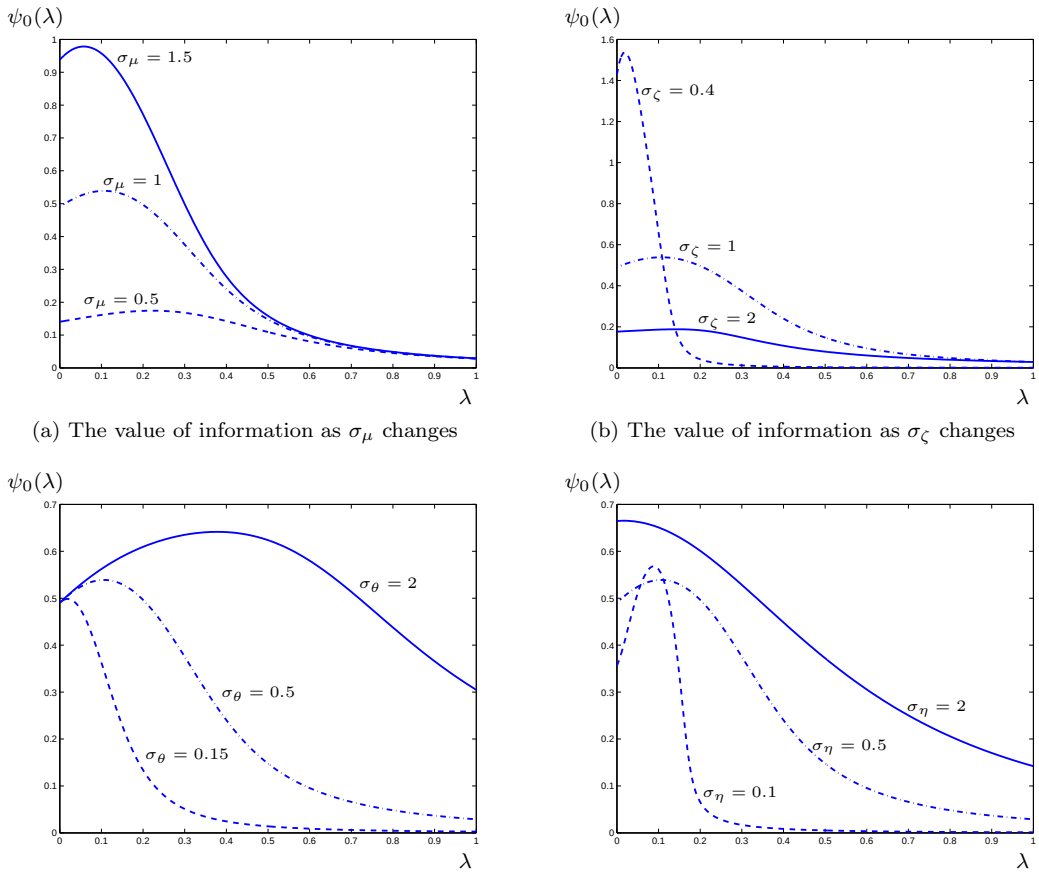
When σ_θ increases information is more valuable because there is higher uncertainty about first-period supply. At the same time, as I have argued above, complementarities in information acquisition are present when first-period supply information is valuable. Therefore, as figure 7c verifies, that higher σ_θ makes first-period supply information more valuable also makes the complementarity more pronounced.

Finally I show comparative statics with respect to uncertainty in second-period supply σ_η in figure 7d. If σ_η increases information is more valuable because there is higher second-period supply risk. Here, the complementarity is less pronounced because the first-period supply $\tilde{\theta}_1$ plays a smaller role in determining capital gains from trade. When σ_η is really high the returns are mostly noise: $\tilde{\eta}$ drowns out the effect of $\tilde{\theta}_1$ in predicting returns.

5.3 Persistent Supply

Apart from the predictability effects that I describe above I show that complementarities in information acquisition *do not arise* under the random-walk assumption for supply. Grossman and Stiglitz (1980) show that in a one-trading-period CARA-normal economy information acquisition presents only substitutabilities. Thus far I have argued that adding another trading period introduces complementarities in information acquisition. But what if the two trading periods were informationally equivalent, in the sense that I could remove one of the two periods without altering how much agents learn? In this case complementarities would disappear. I compare the amount of information in each trading period using precisions delivered by the Kalman filter.

[11] These two effects are also present in the economy of Grossman and Stiglitz (1980).



(a) The value of information as σ_μ changes

(b) The value of information as σ_ζ changes

(c) The value of information as σ_θ changes

(d) The value of information as σ_η changes

Figure 7: Comparative statics on the value-of-information curve over σ_μ, σ_ζ, σ_θ and σ_η when $\rho = 0$. Each figure shows the value of information for high, medium and low levels of a specific uncertainty parameter while holding all other parameters to the baseline $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for high values, dash-dotted curves for medium values and dashed curves for low values of parameters.

Proposition 7 (Learning in the Market)**.** *The increase in precision over time spent in the financial market is characterized by*

$$\frac{1}{Var\big(\tilde{\mu}|\mathcal{F}_2^u\big)} = \frac{1}{Var\big(\tilde{\mu}|\mathcal{F}_1^u\big)} + \left(\frac{p_{\mu\theta} - \rho q_{\mu\theta}}{\sigma_\eta}\right)^2 \tag{10a}$$

and

$$\frac{p_{\mu\theta}^2}{Var\big(\tilde{\theta}_2|\mathcal{F}_2^u\big)} = \frac{q_{\mu\theta}^2}{Var\big(\tilde{\theta}_1|\mathcal{F}_1^u\big)} + \left(\frac{p_{\mu\theta} - \rho q_{\mu\theta}}{\sigma_\eta}\right)^2. \tag{10b}$$

As I have shown in the asset-pricing section of the paper when $\rho < 1$ the learning-adjusted information sensitivity $p_{\mu\theta} - \rho q_{\mu\theta}$ is strictly positive, so this proposition says that how well the uninformed know dividends and supply increases over time spent in the financial market. When $\rho = 1$, however, corollary 2 shows that there is no increase in precision over time. To

understand why, notice that the first-period price contains the information

$$\frac{1}{q_\theta} y_1 = q_{\mu\theta}\tilde{\mu} + \tilde{\theta}_1$$

and the second-period price contains the information

$$\frac{1}{p_\theta} y_2 = p_{\mu\theta}\tilde{\mu} + \tilde{\theta}_1 + \tilde{\eta}.$$

On the one hand, when $\rho = 1$, $p_{\mu\theta} = q_{\mu\theta}$ and $\tilde{\eta}$ is revealed to the uninformed agents at $t = 2$. As a result P_2 contains the *same information* about $\tilde{\mu}$ and $\tilde{\theta}_1$ as P_1 and therefore the uninformed learn nothing about $\tilde{\mu}$ and $\tilde{\theta}_1$ at $t = 2$.[12] On the other hand, when $\rho < 1$, it is the very presence of $\tilde{\eta}$ as an additional source of noise at $t = 2$ that makes P_1 and P_2 diverse signals of $\tilde{\mu}$ over time, which helps the agent increase his precision at $t = 2$. But when supply is a random walk, the first period is informationally redundant. Then the two-trading-period economy is informationally equivalent to a one-trading period economy, which does not exhibit complementarities.

Moreover, under the random-walk-supply assumption the supply channel is not present in ψ_0. The value of information depends on supply in two ways. Firstly, as corollary 3 shows, directly through the relative variance of capital gains from trade. But as the discussion above shows, when $\rho = 1$ observing the difference $P_2 - P_1$ at $t = 2$ does not reveal any information not already known at $t = 1$. In other words, conditional on first-period information, capital gains from trade depend only on $\tilde{\eta}$.[13] Consequently the two agent groups are symmetrically uninformed about $P_2 - P_1$ so that the term measuring its relative variance washes out to one.

Secondly, ψ_0 depends on supply indirectly through the relative hedging behavior of the agents. I show the hedging coefficients for different values of ρ in figure 8. The coefficients β^j are negative because the $t = 1$ return $P_2 - P_1$ is negatively correlated with the $t = 2$ return $D_3 - P_2$. This finding is consistent with Wang (1993) and is line with empirical literature. The informed agents hedge more because having more precise information makes them more confident in taking larger offsetting positions against risk. In addition, $\beta^i = -1$ irrespective of ρ and λ. This is because the only components of $D_3 - P_2$ and $P_2 - P_1$ that the informed do not already know at $t = 1$ are $\tilde{\zeta}$ and $\tilde{\eta}$. The informed cannot hedge these out because they are independent of each other and of every other variable in the economy. Changes in ρ and λ do not influence the distributions of $\tilde{\zeta}$ and $\tilde{\eta}$, therefore such changes do not affect the hedging behavior of the informed agents. For the uninformed, although changes in ρ and λ do not matter for the distributions of $\tilde{\zeta}$ and $\tilde{\eta}$, they do matter for the conditionally joint distribution of $\tilde{\mu}$ and $\tilde{\theta}_1$. But when $\rho = 1$, conditional on first-period information, $P_2 - P_1$ is completely determined by $\tilde{\eta}$, so it co-varies with $D_3 - P_2$ only through $\tilde{\eta}$. As a result the hedging behavior

[12] $\mathrm{Var}(\tilde{\mu}|\mathcal{F}_2^u) = \mathrm{Var}(\tilde{\mu}|\mathcal{F}_1^u)$ from (10a) and $\mathrm{Var}(\tilde{\theta}_1|\mathcal{F}_2^u) = \mathrm{Var}(\tilde{\theta}_2|\mathcal{F}_2^u) - \mathrm{Var}(\tilde{\eta}|\mathcal{F}_2^u) = \mathrm{Var}(\tilde{\theta}_1|\mathcal{F}_1^u)$ from (10b) because $\tilde{\eta}$ is known at $t = 2$. Note, however, that $\tilde{\eta}$ is still unknown at $t = 1$.

[13] Lemma 5 of the appendix shows that conditional on first-period information, when supply is persistent $P_2 - P_1$ is independent of $\tilde{\mu}$ and $\tilde{\theta}_1$.

of the uninformed is the same as that of the informed and therefore $\beta^u = -1$.

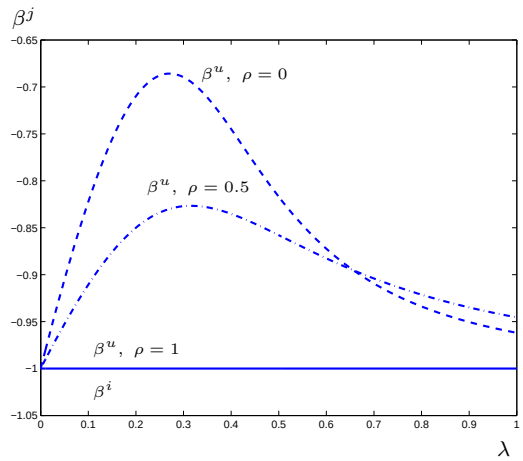


Figure 8: The hedging coefficients β^i and β^u for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$. The hedging coefficient of the informed, β^i, is minus one irrespective of ρ. The two coefficients coincide when $\rho = 1$.

Finally, I establish formally that when supply is extremely persistent information acquisition presents only substitutabilities.

Proposition 8 (No Complementarities When Supply is a Random Walk)**.** *When supply is a Random Walk, the value of information is*

$$e^{2\delta\psi_0} = \frac{\frac{Var(D_3 - P_2|\mathcal{F}_2^u)}{Var(D_3 - P_2|\mathcal{F}_1^u)}}{\frac{Var(D_3 - P_2|\mathcal{F}_2^i)}{Var(D_3 - P_2|\mathcal{F}_1^i)}} \frac{Var\left(D_3 - P_1|\mathcal{F}_1^u\right)}{Var\left(D_3 - P_1|\mathcal{F}_1^i\right)},$$

which is always decreasing in the number of informed agents.

The first term on the right-hand side captures informativeness of prices about liquidating capital gains. The second term is the value of information in an economy starting at $t = 1$ without a trading period at $t = 2$. The capital-gains-from-trade information channel has disappeared from the value of information and so have the complementarities. But more importantly, this result obtains only when supply is extremely persistent. For that to happen, the economic force that causes supply to be persistent must be very strong. Moreover, random-walk supply implies that in a long economy prices are not stationary even if economic fundamentals are stationary. Therefore, in the context of information acquisition in dynamic financial markets, complementarities are not only more economically interesting than substitutabilities, they are also more plausible.

5.4 Further Results

In this section I explore further properties of the economy. First I compare the value of information in my model with that in the one-trading-period model of Grossman and Stiglitz (1980). Consider an economy with a dynamic asset market and suppose supply is independent

over time, $\rho = 0$. Then the amount of liquidity trades is $\tilde{\theta}_1 \sim \mathcal{N}(0, \sigma_\theta^2)$ in the first period and $\tilde{\eta} \sim \mathcal{N}(0, \sigma_\eta^2)$ in the second period. Consider now an economy with a static asset market where the amount of liquidity trades is the total amount of liquidity trades arriving over the horizon of the dynamic asset market, $\tilde{\theta}_1 + \tilde{\eta} \sim \mathcal{N}(0, \sigma_\theta^2 + \sigma_\eta^2)$. One expects that the value of information would be lower in the economy with a dynamic market. This is because in a dynamic market the uninformed receive more information by observing longer price series. As figure 9 shows, however, this is only true for high incidences of informed agents. In contrast, for small numbers of informed agents, the value of information is higher in the economy with a dynamic market. As I have argued above, in dynamic markets informed agents can use supply information to exploit intertemporal liquidity traders. Their superior supply information allows them to do this better than uninformed agents. Moreover, the less competition for liquidity-trader exploitation that informed traders face from other informed traders, the better it is to be informed. This effect is absent from static markets because there is no iteration of trading. This additional feature of dynamic markets, which is strongest when there are few informed agents, makes information more valuable in the dynamic scenario than in the static scenario.

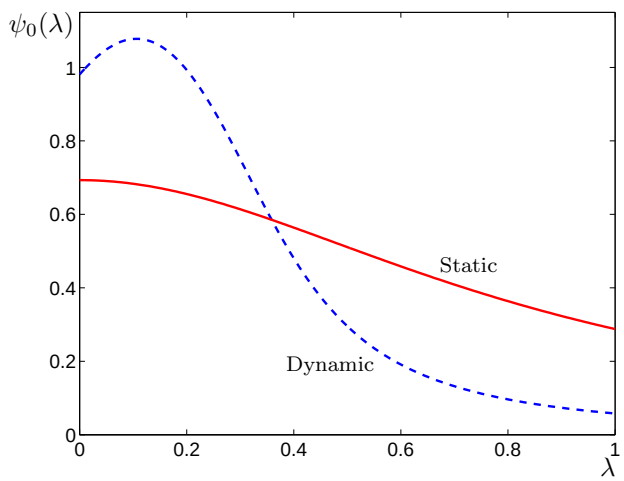
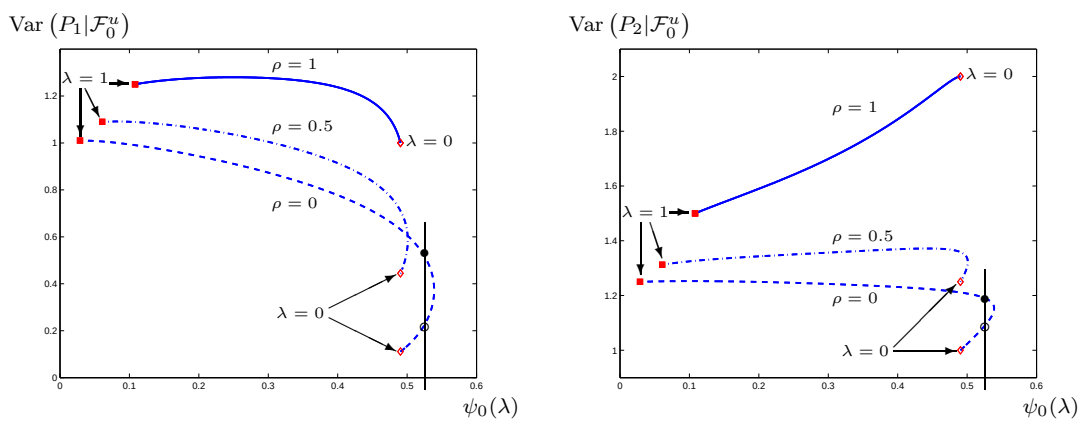


Figure 9: The value of information in a dynamic economy with $\rho = 0$ and liquidity amounts $\sigma_\theta^2 = 1$ and $\sigma_\eta^2 = 1$ (dashed blue curve) compared to the value of information in a static economy with liquidity amount $\sigma_\theta^2 + \sigma_\eta^2$ (solid red curve). The remaining parameters are $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$.

The information complementarity for small numbers of informed agents combined with the information substitutability for large numbers of informed agents gives more than one equilibrium in the information market. For example, as figure 6 shows, when $\rho = 0$ and $\kappa_0 = 0.52$ there are two interior equilibria, at $\lambda_* = 0.04$ and $\lambda^* = 0.17$. I compare such equilibria in terms of risk by examining the ex-ante variance of prices. In figure 10 I plot the variances $\mathrm{Var}\,(P_1|\mathcal{F}_0^u)$ and $\mathrm{Var}\,(P_2|\mathcal{F}_0^u)$ against the value of information as λ varies from zero (empty diamonds) to one (filled squares) for $\rho = 0, 0.5$ and 1. Casual intuition suggests that larger numbers of smart-money investors would stabilize prices and would therefore reduce price variance. But as figures 10a and 10b show, that is not necessarily the case. Higher λ does stabilize prices by making them more informative about dividends. At the same time it also

makes prices riskier for the uninformed because it makes them less informative about capital gains from trade. Moreover, as Wang (1993) explains, higher λ creates an adverse environment for uninformed investors because it means that they have to face more investors with superior information. The result is that price variance can increase in λ, so that when there are two interior equilibria the one with the higher λ can exhibit higher risk.

The presence of more than one equilibrium in the information market also has implications for empirical research. The dynamic model of this study produces a time-series of two price observations. As I show in figure 11, it is possible that two equilibria of equal informational value exhibit the same variance for first-period prices but different variances for second-period prices. In this figure I retrace the price variances of figure 10 for $\rho = 0$ by increasing the supply level uncertainty and keeping all other parameters fixed.[14] Higher uncertainty of supply makes prices overall more risky, but more so for smaller λ than for higher λ. This is because higher numbers of informed agents compensate uninformed agents for losing supply precision by releasing more accurate dividend information. The overall effect is that price variance is not monotonic in λ. This, together with that $\psi_0(\lambda)$ is not monotonic in λ either, gives a variance-versus-value-of-information curve that can cross itself as in figure 11a. In this economy an observer outside the model would find it reasonable to fit a structural break in prices between periods one and two, even though inside the model every agent knows which equilibrium is occurring. This makes the empirical study of asset prices hard because what is a result of equilibrium multiplicity cannot be distinguished from regime switching.

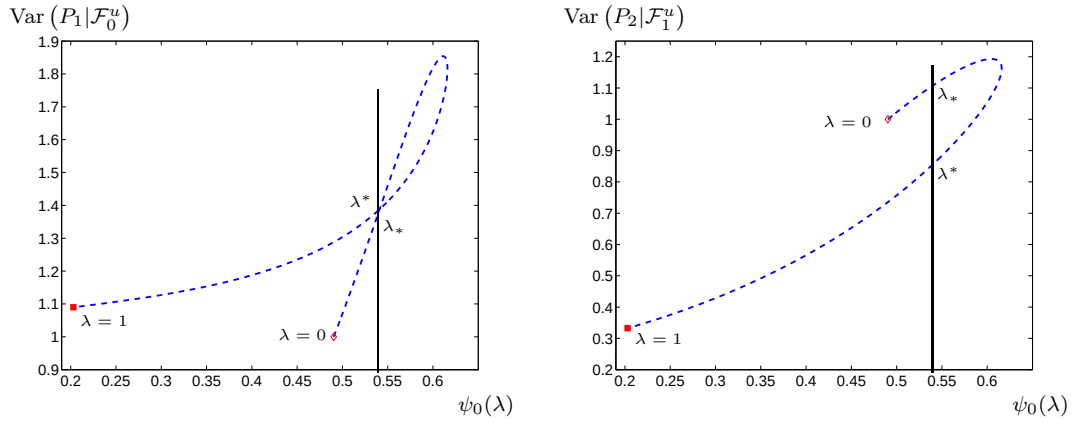


(a) Price variance at $t = 1$ versus the value of information

(b) Price variance at $t = 2$ versus the value of information

Figure 10: Price variances versus the value of information for $\rho = 0, 0.5, 1$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 0.5$ and $\sigma_\eta = 0.5$. Solid curves for $\rho = 1$, dash-dotted curves for $\rho = 0.5$ and dashed curves for $\rho = 0$. Empty diamonds denote $\lambda = 0$ and filled squares denote $\lambda = 1$. When $\kappa_0 = 0.52$ the interior equilibrium with more informed traders (filled circle), is associated with higher uncertainty than the equilibrium with the fewer informed traders (empty circle).

[14] An additional difference from figure 10 is that the second-period price variance of 11b is conditional on first-period information.



(a) First-period price variance versus the value of information

(b) Second-period price variance conditional on first-period information versus the value of information

Figure 11: Price variances versus value of information for $\rho = 0$ with $\delta = 1$, $\sigma_\zeta = 1$, $\sigma_\mu = 1$, $\sigma_\theta = 1.5$ and $\sigma_\eta = 0.5$. Empty diamonds denote $\lambda = 0$ and filled squares denote $\lambda = 1$. When $\kappa_0 = 0.542$ there are two interior equilibria, $\lambda_* = 0.07$ and $\lambda^* = 0.52$ with equal first-period variance but different second-period variances.

6 The Value of Information in Continuous Time

In this section I address how the information-market equilibrium works in an economy with more than two periods by deriving the value of information in the limiting case of continuous time and infinite horizon. A further motivation for this derivation is to connect with existing literature, namely the asset-pricing study of exogenous asymmetric information of Wang (1993). I use that model for the financial market and derive the corresponding equilibrium in the information market. The economy is made up of a continuum of ex-ante identical investors of total mass one. Every investor has preferences of constant absolute risk aversion with coefficient δ. Everyone can invest in a safe bond with constant interest rate r and in a risky stock the dividend process of which is

$$dD_t = \phi_D(\mu_t - D_t)dt + \sigma_D dB_t^D,$$

where B_t^D is a Brownian motion driving the dividends. The growth rate μ_t is not freely observable, but it is known that it follows the process

$$d\mu_t = \phi_\mu(m - \mu_t)dt + \sigma_\mu dB_t^\mu,$$

where B_t^μ is a Brownian motion independent of B_t^D. In the financial market there are λ informed agents that observe the full history of μ_t and $1 - \lambda$ uninformed agents that rely only on prices and dividends to infer as well as possible the value of μ_t. The total stock supply at time t is $1 + \theta_t$, where θ_t is not observable,

$$d\theta_t = -\phi_\theta \theta_t dt + \sigma_\theta dB_t^\theta$$

and B_t^θ is a Brownian motion independent of B_t^μ and B_t^D. In this model Wang (1993) studies the asset pricing implications of exogenous information asymmetry. I endogenize the fraction λ of informed agents as a function of information acquisition costs. I begin with a brief description of the equilibrium in the financial market.

6.1 The Equilibrium in the Financial Market

The equilibrium price process is

$$P_t = p_0 + p_D D_t + p_\mu \mu_t + p_\theta \theta_t + p_{\hat{\mu}} \hat{\mu}_t + p_{\hat{\theta}} \hat{\theta}_t. \tag{11}$$

The information that the informed have is the complete history of the dividend D_t, of the price P_t and of the dividend growth rate μ_t. Let the σ-algebra $\mathcal{F}_t^i$ represent this information. The information $\mathcal{F}_t^u$ that the uninformed have at time t is the complete history of the dividend D_t and the price P_t only. The inferences $\hat{\mu}_t$ and $\hat{\theta}_t$ are the best estimates of μ_t and θ_t given the information $\mathcal{F}_t^u$,

$$\hat{\mu}_t = \mathbb{E}\left[\mu_t \middle| \mathcal{F}_t^u\right] = \mathbb{E}\left[\mu_t \middle| \{D_s, P_s\}_{0 \le s \le t}\right]$$

and

$$\hat{\theta}_t = \mathbb{E}\left[\theta_t \middle| \mathcal{F}_t^u\right] = \mathbb{E}\left[\theta_t \middle| \{D_s, P_s\}_{0 \le s \le t}\right].$$

These estimates are given by a Kalman-Bucy filtering, with steady-state solution

$$d\begin{pmatrix} \hat{\mu}_t \\ \hat{\theta}_t \end{pmatrix} = \begin{pmatrix} \phi_\mu (m - \hat{\mu}_t) \\ -\phi_\theta \hat{\theta}_t \end{pmatrix} dt + h(q_{xx})^{\frac{1}{2}} d\hat{B}_t$$

The filtering innovation $\hat{B}_t$ is a two-dimensional vector Brownian motion. The matrices q_{xx} and h are constants provided in terms of the model parameters in the appendix. These estimates are also available to the informed agents. Therefore the informed investors observe, either directly or by inference, all other quantities present in the price process apart from the noisy supply θ_t. But they also observe the price, therefore the supply θ_t is revealed to them fully.

6.1.1 Agent Demand

The demand of each agent group is the solution to a portfolio problem taking the returns in excess of the risk-free rate, R_t, as given. The excess returns follow the process

$$dR_t = (D_t - rP_t)dt + dP_t. \tag{12}$$

The only quantities that matter for portfolio selection are those that help predict excess returns. As Wang (1993) shows the state vector for the uninformed can be reduced to the column vector $S_t^u = (1 \ \hat{\theta}_t)^T$ and the state vector for the informed can be reduced to the column vector $S_t^i = (1 \ \theta_t \ \hat{\mu}_t - \mu_t)^T$. This reduction also gives explicit expressions for p_D and $p_\mu + p_{\hat{\mu}}$. Consider

now the portfolio problem of an investor in group j for $j = i, u$. At time t the dollar amount invested in the stock is X_t^j, the wealth is W_t^j and consumption is c_t^j. Let ν be the discount factor. The portfolio selection problem is

$$\begin{aligned}
\max_{\{c_s^j, X_s^j\}_{t \le s \le \infty}} \quad & \mathbb{E}\left[\int_t^\infty e^{-\nu s}\left(-e^{-\delta c_s^j}\right) ds \middle| \mathcal{F}_t^j\right] \\
\text{s.t. } dW_t^j &= \left(rW_t^j - c_t^j\right) dt + X_t^j dR_t \\
dR_t &= m_R^j S_t^j dt + v_R^j d\hat{B}_t \\
dS_t^j &= m_S^j S_t^j dt + v_S^j d\hat{B}_t
\end{aligned}$$

where m_S^j, v_S^j, m_R^j and v_R^j are constant matrices. It is well known that the value function for this type of problem is separable in time, state variables and wealth. It has the form $e^{-\nu t} J^j(W_t^j, S_t^j)$ where

$$J^j(W_t^j, S_t^j) = -A^j e^{-r\delta W_t^j - \frac{1}{2} S_t^{jT} \gamma^j S_t^j}.$$

The constant A^j is scalar and γ^j is a matrix of constants characterized in the appendix.

6.1.2 Market Clearing

A fraction λ of agents are informed. The informed agents' demand of the stock is $X_t^i = d^i S_t^i$. Here d^i is a 1×3 row vector that depends on the coefficients of the price process and the parameters of the economy. Similarly the uninformed agents' demand of the stock is $X_t^u = d^u S_t^u$ where d^u is a 1×2 row vector. I give d^i and d^u in the appendix. The stock market clears when the aggregate investor demand equals the noisy supply,

$$\lambda X_t^i + (1 - \lambda) X_t^u = 1 + \theta_t.$$

Matching coefficients in the underlying state variables of the two agent groups gives three non-linear equations. These pin down the coefficients in the price function in terms of λ.

6.2 The Equilibrium in the Information Market

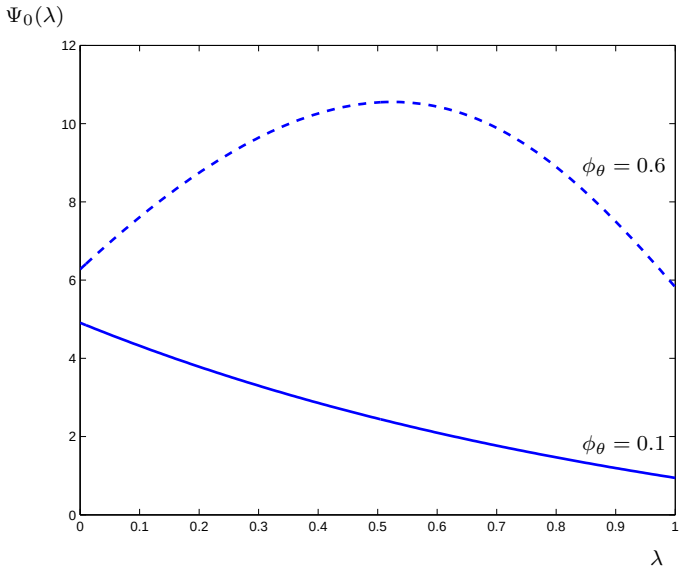
Next I endogenize the fraction of informed agents by using the same equilibrium concept as in definition 2, where the financial equilibrium is now a sequence of prices $\{P_t(\lambda)\}_{0 \le t \le \infty}$ given by the financial market equilibrium of Wang (1993) that I have just described. Each agent has the option to subscribe to the full observations of μ_t at the beginning of time by incurring the cost κ_0. After $t = 0$ the agents cannot change their information status. The equilibrium number of informed agents λ^* is determined in the same way as in discrete time, so here I focus on describing the derivation of the value of information. The value of information $\Psi_0(\lambda)$ is defined as

$$J^u\left(W_0, S_0^u; \lambda\right) = \mathbb{E}\left[J^i\left(W_0 - \Psi_0(\lambda), S_0^i; \lambda\right) \middle| \mathcal{F}_0^u\right]$$

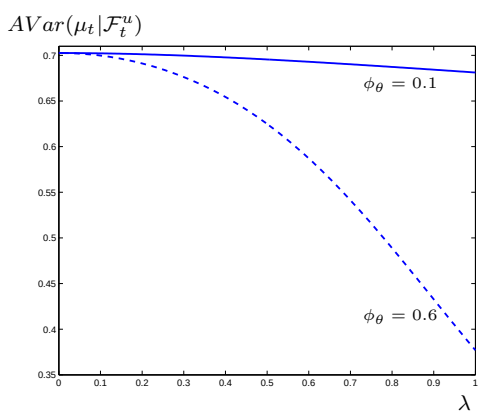
The calculation of $\Psi_0(\lambda)$ is in closed form in terms of the solution of the portfolio problems and the price coefficients. Because, however, the price coefficients have to be solved numerically, I do not have a completely closed form of $\Psi_0(\lambda)$ in terms of the model parameters. Similarly to section 4, the value of information can be written as

$$\Psi_0(\lambda) = \frac{1}{r\delta} \log \left(\frac{J^u(W_0, S_0^u; \lambda)}{\mathbb{E}\left[J^i(W_0, S_0^i; \lambda) | \mathcal{F}_0^u \right]} \right).$$

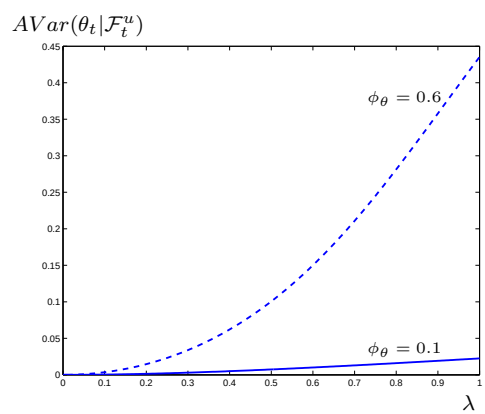
The result of this calculation depends on the uninformed agents' prior supply estimate, $\hat{\theta}_0$. I set $\hat{\theta}_0$ equal to the long-run mean of the θ process, which is zero. This is the same as requiring that the uninformed prior is specified correctly.



(a) The value of information in continuous time



(b) The asymptotic error of inference in μ_t



(c) The asymptotic error of inference in θ_t

Figure 12: Value of information and asymptotic error variances for $\phi_\theta = 0.1$ and 0.6 with $\delta = 3$, $r = 0.05$, $\phi_D = 0.4$, $\sigma_D = 1$, $m = 0.8$, $\phi_\mu = 0.2$, $\sigma_\mu = 0.6$ and σ_θ such that $\sigma_\theta^2/2\phi_\theta = 1$. Solid curves for $\phi_\theta = 0.1$ and dashed curves for $\phi_\theta = 0.6$.

In figure 12 I show the value of information $\Psi_0(\lambda)$ and the associated asymptotic inference

qualities of dividend forecast and supply. As more informed agents enter the economy prices become more informative about dividends but less informative about supply, as witnessed by that $AVar(\mu_t|\mathcal{F}_t^u)$ decreases in λ and that $AVar(\theta_t|\mathcal{F}_t^u)$ increases in λ. The question now is, when do complementarities arise? A calculation based on matching the correlation structure of an AR(1) process with that of the Ornstein-Uhlenbeck process shows that $\rho = e^{-\phi_\theta \Delta t}$, where Δt is the time between consecutive periods of realizations of the AR(1) process. Therefore complementarities should arise when ϕ_θ is high. As the top plot of figure 12 verifies, when ϕ_θ is high, that $AVar(\theta_t|\mathcal{F}_t^u)$ is increasing in λ makes the value of information non-monotonic in λ.

Finally let me address the differences between the model in discrete time and the model in continuous time. One advantage of the two-period model is that I can have an expression for the value of information in terms of conditional moments of returns. At the time of this writing, the best available description of the value of information in continuous time is as a ratio of value functions. Moreover, that the continuous-time model is in steady state does not allow for the explicit expression of learning-over-time effects which, as evident in proposition 7, are quite important. But perhaps most importantly, the economic forces of the continuous-time model are clearly expressed already in the two-period model. This says that the length of the economy does not matter for the value of information. To gain some intuition about why this is the case, consider the following argument. Suppose that the economy was in continuous time but that the world ended at a random date τ. To keep things simple further suppose that τ was determined by the arrival of a Poisson shock of rate ν^τ, independently of everything else in the economy. By a standard exchange-of-integrals argument the objective value function of each agent group j would be

$$\mathbb{E}\left[\int_t^\tau e^{-\nu s}\left(-e^{-\delta c_s^j}\right) ds \middle| \mathcal{F}_t^j\right] = \mathbb{E}\left[\int_t^\infty e^{-(\nu+\nu^\tau)s}\left(-e^{-\delta c_s^j}\right) ds \middle| \mathcal{F}_t^j\right].$$

That is, the only change in the economy is that the discount factor ν has increased by ν^τ. But as the appendix shows, the discount factor washes out completely in the expression for $\Psi_0(\lambda)$. Therefore in this context the length of the economy does not matter for the value of information.

7 Conclusion

To conclude, supply is valuable information in dynamic financial markets because it captures information about short-term capital gains. When the persistence of supply is low the level of supply determines changes in supply and thus supply drives changes in prices. This implies that uninformed investors use prices to learn information about two quantities, dividend information and supply. In this joint estimation dividends and supply act as noise with respect to each other. As a result, changes in the information market make the estimation qualities of dividends and supply move in opposite directions.

This trade-off in estimation creates a similar trade-off in informativeness of prices about dividends and capital gains. As more informed agents enter the economy they make prices more informative about cash flows, but less informative about short-term changes in prices. The combination of the two effects gives a value of information that is not monotonic in the number of informed agents. This phenomenon creates multiple equilibria in the information market and makes prices fragile in perturbations of information costs.

This price fragility is often interpreted as a source of "structural breaks", or "regime switching" in prices. The intuition is that if the information decision was to be repeated over time, then the multiplicity of equilibria would translate to time-varying moments of prices. In order to argue this point further the iteration of the information decision must be part of the model. Moreover, in reality markets are dynamic environments. Because asset markets and information markets interact, if we want to understand either market we must understand intertemporal information acquisition. The dynamic model of this paper provides a foundation for such studies.

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A Appendix

A.1 Auxiliary Results

Lemma 1. *For two σ-algebrae $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ is contained in $\mathcal{H}_2$ and the normal random variables Z and W,*

$$Cov\left(\mathbb{E}\left[Z|\mathcal{H}_2\right], \mathbb{E}\left[W|\mathcal{H}_2\right]|\mathcal{H}_1\right) = Cov\left(Z, W|\mathcal{H}_1\right) - Cov\left(Z, W|\mathcal{H}_2\right).$$

Proof. The law of total covariance states that for the random variables X and Y, conditional on the σ-algebra $\mathcal{G}$,

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[\mathrm{Cov}(X, Y|\mathcal{G})\right] + \mathrm{Cov}(\mathbb{E}[X|\mathcal{G}], \mathbb{E}[Y|\mathcal{G}]).$$

When X and Y are normal the conditional covariance is constant, therefore applying this to the normal random variables $X = \mathbb{E}\left[Z|\mathcal{H}_2\right]$ and $Y = \mathbb{E}\left[W|\mathcal{H}_2\right]$ and the σ-algebra $\mathcal{H}_1$ I get

$$\begin{aligned}\mathrm{Cov}\left(\mathbb{E}\left[Z|\mathcal{H}_2\right], \mathbb{E}\left[W|\mathcal{H}_2\right]|\mathcal{H}_1\right) &= \mathrm{Cov}(\mathbb{E}\left[Z|\mathcal{H}_2\right], \mathbb{E}\left[W|\mathcal{H}_2\right]) - \mathrm{Cov}(\mathbb{E}\left[Z|\mathcal{H}_1\right], \mathbb{E}\left[W|\mathcal{H}_1\right]) \\ &= \mathrm{Cov}\left(Z, W|\mathcal{H}_1\right) - \mathrm{Cov}\left(Z, W|\mathcal{H}_2\right)\end{aligned}$$

by the law of iterated expectations and the law of total covariance. □

Lemma 2. *For two random variables $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ where $Cov(X_1, X_2) = \sigma_{12}$,*

$$\begin{aligned}&\mathbb{E}\big[\exp\{b_1X_1 + b_2X_2 + a_{11}X_1^2 + 2a_{12}X_1X_2 + a_{22}X_2^2\}\big] = \\ &\frac{1}{S^{\frac{1}{2}}}\exp\Bigg\{\frac{1}{S}\Bigg\{\frac{1}{2}\left[b_1^2\left(\sigma_1^2 - 2a_{22}|\Sigma|\right) + 2b_1b_2\left(\sigma_{12} + 2a_{12}|\Sigma|\right) + b_2^2\left(\sigma_2^2 - 2a_{11}|\Sigma|\right)\right] \\ &\qquad +\mu_1\left[b_1 + 2(a_{11}b_2 - a_{12}b_1)\sigma_{12} + 2(a_{12}b_2 - a_{22}b_1)\sigma_2^2\right] \\ &\qquad +\mu_2\left[b_2 + 2(a_{12}b_1 - a_{11}b_2)\sigma_1^2 + 2(a_{22}b_1 - a_{12}b_2)\sigma_{12}\right] \\ &\qquad +\mu_1^2a_{11}\left(1 - 2a_{22}\sigma_2^2\right) + 2\mu_1\mu_2\left(a_{12} + 2|A|\sigma_{12}\right) + \mu_2^2a_{22}\left(1 - 2a_{11}\sigma_1^2\right)\Bigg\}\Bigg\}\end{aligned}$$

where

$$\begin{aligned}S &= |I - 2\Sigma A| = 1 - 2(a_{11}\sigma_1^2 + 2a_{12}\sigma_{12} + a_{22}\sigma_2^2) + 4|A||\Sigma| \\ |A| &= a_{11}a_{22} - a_{12}^2 \\ |\Sigma| &= \sigma_1^2\sigma_2^2 - \sigma_{12}^2\end{aligned}$$

Proof. This is a special case of a standard property of the multivariate normal distribution for two random variables, see for example Vives (2008) section 10.2.4 and references therein. □

A.2 Results on the Model

Kalman Filter (Solution)**.** The first-period inferences of the uninformed are

$$\begin{aligned}\hat{\mu}_1 &= \frac{q_{\mu\theta}\sigma_\mu^2}{q_{\mu\theta}^2\sigma_\mu^2 + \sigma_\theta^2}\left(q_{\mu\theta}\tilde{\mu} + \tilde{\theta}_1\right), \\ \hat{\theta}_1 &= \frac{\sigma_\theta^2}{q_{\mu\theta}^2\sigma_\mu^2 + \sigma_\theta^2}\left(q_{\mu\theta}\tilde{\mu} + \tilde{\theta}_1\right).\end{aligned}$$

The first-period inference qualities are

$$\begin{aligned}\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right) &= \frac{\sigma_\mu^2\sigma_\theta^2}{q_{\mu\theta}^2\sigma_\mu^2 + \sigma_\theta^2}, \\ \mathrm{Cov}\left(\tilde{\mu}, \tilde{\theta}_1|\mathcal{F}_1^u\right) &= -q_{\mu\theta}\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right),\end{aligned}$$

$$\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right) = q_{\mu\theta}^2 \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right).$$

Forecast:

$$\mathbb{E}\left[\tilde{\mu}|\mathcal{F}_1^u\right] = \hat{\mu}_1 \quad (\tilde{\mu} \text{ does not change})$$
$$\mathbb{E}\left[\tilde{\theta}_2|\mathcal{F}_1^u\right] = \rho\hat{\theta}_1$$

The second-period inferences of the uninformed are

$$\hat{\mu}_2 = \hat{\mu}_1 + (p_{\mu\theta} - \rho q_{\mu\theta})\frac{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right)}{\sigma_\eta^2}\left[p_{\mu\theta}(\tilde{\mu} - \hat{\mu}_1) + \tilde{\theta}_2 - \rho\hat{\theta}_1\right],$$
$$\hat{\theta}_2 = \rho\hat{\theta}_1 + \left[\frac{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right)}{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)} - \rho q_{\mu\theta}(p_{\mu\theta} - \rho q_{\mu\theta})\frac{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right)}{\sigma_\eta^2}\right]\left[p_{\mu\theta}(\tilde{\mu} - \hat{\mu}_1) + \tilde{\theta}_2 - \rho\hat{\theta}_1\right].$$

The second-period inference qualities are

$$\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right) = \frac{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)\sigma_\eta^2}{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)\left(p_{\mu\theta} - \rho q_{\mu\theta}\right)^2 + \sigma_\eta^2},$$
$$\mathrm{Cov}\left(\tilde{\mu}, \tilde{\theta}_2|\mathcal{F}_2^u\right) = -p_{\mu\theta}\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right),$$
$$\mathrm{Var}\left(\tilde{\theta}_2|\mathcal{F}_2^u\right) = p_{\mu\theta}^2\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right).$$

□

Proof of Proposition 1. The value function of each agent group $j = i, u$ is

$$J^j(W_0) = \max_{x_1^j, x_2^j}\left\{-e^{-\delta W_0}e^{\delta x_1^j P_1}\mathbb{E}\left[e^{-\delta x_1^j P_2 + \delta x_2^j P_2}\mathbb{E}\left[e^{-\delta x_2^j D_3}\middle|\mathcal{F}_2^j\right]\middle|\mathcal{F}_1^j\right]\right\}$$

Define

$$E_2\left(x_2; \mathcal{F}_2^j\right) = e^{\delta x_2 P_2}\mathbb{E}\left[e^{-\delta x_2 D_3}\middle|\mathcal{F}_2^j\right]$$
$$E_1\left(x_1; \mathcal{F}_1^j\right) = e^{\delta x_1 P_1}\mathbb{E}\left[e^{-\delta x_1 P_2}\min_{x_2^j}\left\{E_2^j\left(x_2^j; \mathcal{F}_2^j\right)\right\}\middle|\mathcal{F}_1^j\right]$$

The objective function of the innermost optimization is

$$E_2\left(x_2; \mathcal{F}_2^j\right) = e^{-\delta x_2\left(E\left[D_3|\mathcal{F}_2^j\right] - P_2\right) + \frac{1}{2}x_2^2\mathrm{Var}\left(D_3|\mathcal{F}_2^j\right)},$$

so the first-order condition gives

$$x_2^{j*} = \frac{\mathbb{E}\left[D_3|\mathcal{F}_2^j\right] - P_2}{\delta\mathrm{Var}\left(D_3|\mathcal{F}_2^j\right)} = \frac{\mathbb{E}\left[D_3 - P_2|\mathcal{F}_2^j\right]}{\delta\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^j\right)},$$

which establishes (i). Plugging the first-order condition back into the value function shows that the objective function of the outermost optimization is

$$E_1\left(x_1; \mathcal{F}_1^j\right) = \mathbb{E}\left[e^{-\delta x_1^j(P_2 - P_1) - \frac{\mathbb{E}^2\left[D_3 - P_2|\mathcal{F}_2^j\right]}{2\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^j\right)}}\middle|\mathcal{F}_1^j\right].$$

To carry out the calculation of this expectation, apply lemma 2 with $X_1 = P_2 - P_1$ and $X_2 =$

$\mathbb{E}\left[D_3 - P_2|\mathcal{F}_2^j\right]$ conditionally on $\mathcal{F}_1^j$, i.e. with

$$\begin{aligned}
\mu_1 &= \mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right] \\
\mu_2 &= \mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right] \\
\sigma_1^2 &= \mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right) \\
\sigma_2^2 &= \mathrm{Var}\left(\mathbb{E}[D_3 - P_2|\mathcal{F}_2^j]|\mathcal{F}_1^j\right) \\
\sigma_{12} &= \mathrm{Cov}\left(P_2 - P_1, \mathbb{E}[D_3 - P_2|\mathcal{F}_2^j]|\mathcal{F}_1^j\right)
\end{aligned}$$

and $b_1 = -\delta x_1^j$, $b_2 = 0$, $a_{11} = a_{12} = 0$, $a_{22} = -\frac{1}{2\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^j\right)}$. Then $|A| = 0$ and $S = 1 - 2a_{22}\sigma_2^2$ so that the expectation is

$$E_1\left(x_1; \mathcal{F}_1^j\right) = \frac{1}{S^{\frac{1}{2}}} \exp\left\{\frac{1}{S}\left\{\frac{1}{2}\delta^2 {x_1}^2\left(\sigma_1^2 - 2a_{22}|\Sigma|\right) - \delta x_1\left[(1 - 2a_{22}\sigma_2^2)\mu_1 + 2a_{22}\sigma_{12}\mu_2\right] + \mu_2^2 a_{22}\right\}\right\}.$$

The first-order condition at $t = 1$ gives

$$x_1^{j\,*} = \frac{(1 - 2a_{22}\sigma_2^2)\mu_1 + 2a_{22}\sigma_{12}\mu_2}{\delta\left(\sigma_1^2 - 2a_{22}|\Sigma|\right)} = \frac{\left(\sigma_2^2 - \frac{1}{2a_{22}}\right)\mu_1 - \sigma_{12}\mu_2}{\delta\left(|\Sigma| - \frac{1}{2a_{22}}\sigma_1^2\right)}.$$

Using lemma 1 I obtain

$$\begin{aligned}
\sigma_{12} &= \mathrm{Cov}\left(P_2 - P_1, D_3 - P_2|\mathcal{F}_1^j\right), \\
\sigma_2^2 - \frac{1}{2a_{22}} &= \mathrm{Var}\left(\mathbb{E}[D_3 - P_2|\mathcal{F}_2^j]|\mathcal{F}_1^j\right) + \mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^j\right) = \mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right), \\
|\Sigma| - \frac{1}{2a_{22}}\sigma_1^2 &= \sigma_1^2\left(\sigma_2^2 - \frac{1}{2a_{22}}\right) - \sigma_{12}^2.
\end{aligned}$$

Now define

$$h^j = \frac{\mathrm{Cov}\left(D_3 - P_2, P_2 - P_1\middle|\mathcal{F}_1^j\right)}{\mathrm{Var}(D_3 - P_2|\mathcal{F}_1^j)},$$

so that the first-order condition at $t = 1$ becomes

$$x_1^{j\,*} = \frac{\mu_1 - h^j\mu_2}{\delta\left(\sigma_1^2 - \frac{\sigma_{12}^2}{\mathrm{Var}(D_3 - P_2|\mathcal{F}_1^j)}\right)}.$$

Moreover,

$$\begin{aligned}
\sigma_1^2 - \frac{\sigma_{12}^2}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)} &= \mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right) - h^j\mathrm{Cov}\left(D_3 - P_2, P_2 - P_1\middle|\mathcal{F}_1^j\right) \\
&= \mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)
\end{aligned}$$

which establishes (ii). Finally, define K_1^j to be the inverse of S, then

$$K_1^j = \frac{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)},$$

which is an implication of lemma 1. □

Remark 1.

$$\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right) = \mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right) - \left(h^j\right)^2 \mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right).$$

Proof of Corollary 1. The result follows by plugging the definition of β^j into the first-period demand decomposition and carrying out the algebra. □

Proposition 9 (Equilibrium Prices)**.** *In equilibrium, for a fixed fraction of informed investors λ,*

(i) The second-period price coefficients are

$$\begin{aligned} p_\mu &= \lambda \frac{\Pi_2^i}{\Pi_2} \\ p_{\hat{\mu}} &= 1 - p_\mu \\ p_\theta &= -\frac{\delta}{\Pi_2} \\ p_{\hat{\theta}} &= 0 \end{aligned}$$

(ii) The first-period price coefficients are given by

$$\begin{aligned} q_\mu &= \lambda \frac{\Pi_1^i}{\Pi_1}\left[1 - \left(1 + h^i\right) p_{\hat{\mu}} \frac{Var(\tilde{\mu}|\mathcal{F}_2^u)}{Var(\tilde{\mu}|\mathcal{F}_1^u)}\right] \\ q_\theta &= \lambda \frac{\Pi_1^i}{\Pi_1}\left(1 + h^i\right) \rho p_\theta - \frac{\delta}{\Pi_1} \\ q_{\hat{\mu}} &= 1 - q_\mu \\ q_{\hat{\theta}} &= (1-\lambda)\frac{\Pi_1^u}{\Pi_1}\left(1 + h^u\right) \rho p_\theta \end{aligned}$$

The second-period precisions Π_2^i, Π_2^u and Π_2, the first-period precisions Π_1^i, Π_1^u and Π_1 and the hedging coefficients h^i and h^u are given in terms of $p_{\mu\theta}$ and $q_{\mu\theta}$ in Lemma 3.

Proof of Propositions 2 and 9. Using second-period optimal demands and equation (1b), matching coefficients in second-period market clearing gives

$$\Pi_2 p_\mu = \lambda \Pi_2^i \tag{13a}$$
$$\Pi_2 p_\theta = -\delta \tag{13b}$$
$$\Pi_2 p_{\hat{\mu}} = (1-\lambda)\Pi_2^u \tag{13c}$$
$$p_{\hat{\theta}} = 0. \tag{13d}$$

Moreover, note that

$$p_\mu + p_{\hat{\mu}} = 1.$$

This establishes B.(i). Now divide (13a) by (13b) to get

$$p_{\mu\theta} = -\frac{\lambda \Pi_2^i}{\delta} = -\frac{\lambda}{\delta \sigma_\zeta^2},$$

which establishes A.(i). Next, notice that $\mathbb{E}\left[D_3|\mathcal{F}_1^i\right] = \tilde{\mu}$, $\mathbb{E}\left[D_3|\mathcal{F}_1^u\right] = \hat{\mu}_1$ and that from proposition 7

$$\mathbb{E}\left[P_2|\mathcal{F}_1^i\right] = \bar{p}_\mu \tilde{\mu} + \bar{p}_{\hat{\mu}} \hat{\mu}_1 + \rho p_\theta \tilde{\theta}_1$$

where

$$\begin{aligned}
\bar{p}_\mu &= p_\mu + p_{\hat{\mu}} \frac{\sigma_\mu^2 \sigma_\theta^2 (p_{\mu\theta} - \rho q_{\mu\theta})^2}{\sigma_\mu^2 \sigma_\theta^2 (p_{\mu\theta} - \rho q_{\mu\theta})^2 + \sigma_\eta^2 (q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2)} \\
&= p_\mu + p_{\hat{\mu}} (p_{\mu\theta} - \rho q_{\mu\theta})^2 \frac{\mathrm{Var}\,(\tilde{\mu}|\mathcal{F}_2^u)}{\sigma_\eta^2} \\
\bar{p}_{\hat{\mu}} &= p_{\hat{\mu}} \frac{\sigma_\eta^2 (q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2)}{\sigma_\mu^2 \sigma_\theta^2 (p_{\mu\theta} - \rho q_{\mu\theta})^2 + \sigma_\eta^2 (q_{\mu\theta}^2 \sigma_\mu^2 + \sigma_\theta^2)} = 1 - \bar{p}_\mu \\
&= p_{\hat{\mu}} \frac{\mathrm{Var}\,(\tilde{\mu}|\mathcal{F}_2^u)}{\mathrm{Var}\,(\tilde{\mu}|\mathcal{F}_1^u)}
\end{aligned}$$

Using iterated conditional expectations I get

$$\mathbb{E}\left[P_2|\mathcal{F}_1^u\right] = \hat{\mu}_1 + \rho p_\theta \hat{\theta}_1.$$

The above conditional expectations, first-period optimal demands, equation (1a) and matching coefficients in first-period market clearing gives

$$\Pi_1 q_\mu = \lambda \Pi_1^i \left[\left(1 + h^i\right) \bar{p}_\mu - h^i\right] \tag{14a}$$

$$\Pi_1 q_\theta = \lambda \Pi_1^i \left(1 + h^i\right) \rho p_\theta - \delta \tag{14b}$$

$$\Pi_1 q_{\hat{\mu}} = \lambda \Pi_1^i \left(1 + h^i\right) \bar{p}_{\hat{\mu}} + (1 - \lambda) \Pi_1^u \tag{14c}$$

$$\Pi_1 q_{\hat{\theta}} = (1 - \lambda) \Pi_1^u \left(1 + h^u\right) \rho p_\theta \tag{14d}$$

where I note that

$$q_\mu + q_{\hat{\mu}} = 1.$$

The last five equations together with proposition 7 establish B.(ii). Now divide (14a) by (14b) to get

$$q_{\mu\theta} = \frac{\lambda \Pi_1^i \left[\left(1 + h^i\right) \bar{p}_\mu - h^i\right]}{\lambda \Pi_1^i \left(1 + h^i\right) \rho p_\theta - \delta}$$

Rearranging this and carrying out the algebra gives the polynomial in A.(ii). □

Lemma 3 (Auxiliary Quantities in Asset Pricing)**.** *The second-period precisions are given by*

$$\begin{aligned}
\left(\Pi_2^i\right)^{-1} &= \sigma_\zeta^2 \\
(\Pi_2^u)^{-1} &= \sigma_\zeta^2 + Var(\tilde{\mu}|\mathcal{F}_2^u) \\
\Pi_2 &= \lambda \Pi_2^i + (1 - \lambda) \Pi_2^u.
\end{aligned}$$

The first-period precisions are given by

$$\begin{aligned}
\left(\Pi_1^i\right)^{-1} &= \left(1 + h^i\right)^2 {\chi_{\theta 2}}^2 \sigma_\eta^2 + {h^i}^2 \sigma_\zeta^2 \\
(\Pi_1^u)^{-1} &= (1 + h^u)^2 {\chi_{\theta 2}}^2 \sigma_\eta^2 + {h^u}^2 \sigma_\zeta^2 \\
&\quad + \left[(1 + h^u) \chi_\mu - h^u\right]^2 Var(\tilde{\mu}|\mathcal{F}_1^u) + (1 + h^u)^2 \left(\rho \chi_{\theta 2} + \chi_{\theta 1}\right)^2 Var\left(\tilde{\theta}_1|\mathcal{F}_1^u\right) \\
&\quad + 2\left[(1 + h^u) \chi_\mu - h^u\right] (1 + h^u) \left(\rho \chi_{\theta 2} + \chi_{\theta 1}\right) Cov\left(\tilde{\mu}, \tilde{\theta}_1|\mathcal{F}_1^u\right) \\
\Pi_1 &= \lambda \Pi_1^i + (1 - \lambda) \Pi_1^u.
\end{aligned}$$

Here,

$$h^i = -\frac{{\chi_{\theta 2}}^2 \sigma_\eta^2}{\sigma_\zeta^2 + {\chi_{\theta 2}}^2 \sigma_\eta^2}$$

$$
\begin{aligned}
h^u = \Big\{ & -\chi_{\theta 2}{}^2\sigma_\eta^2 + \chi_\mu(1-\chi_\mu)\, Var(\tilde{\mu}|\mathcal{F}_1^u) - (\rho\chi_{\theta 2} + \chi_{\theta 1})^2\ Var\left(\tilde{\theta}_1|\mathcal{F}_1^u\right) \\
& + (1-2\chi_\mu)\,(\rho\chi_{\theta 2} + \chi_{\theta 1})\ Cov\left(\tilde{\mu}, \tilde{\theta}_1|\mathcal{F}_1^u\right) \Big\} \\
\times \Big\{ & \sigma_\zeta^2 + \chi_{\theta 2}{}^2\sigma_\eta^2 + (1-\chi_\mu)^2\, Var(\tilde{\mu}|\mathcal{F}_1^u) + (\rho\chi_{\theta 2} + \chi_{\theta 1})^2\ Var\left(\tilde{\theta}_1|\mathcal{F}_1^u\right) \\
& - 2\,(1-\chi_\mu)\,(\rho\chi_{\theta 2} + \chi_{\theta 1})\ Cov\left(\tilde{\mu}, \tilde{\theta}_1|\mathcal{F}_1^u\right) \Big\}^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_\mu &= 1 - p_{\hat{\mu}} \frac{Var(\tilde{\mu}|\mathcal{F}_2^u)}{\sigma_\mu^2} \\
\rho\chi_{\theta 2} + \chi_{\theta 1} &= \rho p_\theta + p_{\hat{\mu}} q_{\mu\theta} \frac{Var(\tilde{\mu}|\mathcal{F}_2^u)}{\sigma_\theta^2} \\
\chi_{\theta 2} &= p_\theta + p_{\hat{\mu}}\,(p_{\mu\theta} - \rho q_{\mu\theta})\, \frac{Var(\tilde{\mu}|\mathcal{F}_2^u)}{\sigma_\eta^2}.
\end{aligned}
$$

The error variances and covariances are given in terms of $p_{\mu\theta}$ and $q_{\mu\theta}$ in the solution to the Kalman Filter above.

Proof. These expressions follow by omitted algebraic manipulations. □

Proof of Proposition 3.

(i) A financial market equilibrium exists when equation (5) has a real root. Carrying out the algebra shows that (5) can be written as

$$
\gamma_3 q_{\mu\theta}^3 + \gamma_2 q_{\mu\theta}^2 + \gamma_1 q_{\mu\theta} + \gamma_0 = 0, \tag{6}
$$

where

$$
\begin{aligned}
\gamma_3 &= \sigma_\mu^2 \left(\rho p_{\mu\theta} - \delta\sigma_\eta^2\right)\left(\rho^2\sigma_\theta^2 + \sigma_\eta^2\right) \\
\gamma_2 &= \frac{\sigma_\theta^2}{\sigma_\zeta^2}\left\{-\sigma_\zeta^2(\sigma_\eta^2 + 3\rho^2\sigma_\theta^2)p_{\mu\theta}^2 + \delta\sigma_\zeta^2\sigma_\eta^2\left(\sigma_\eta^2 + \rho\,(1+\rho)\,\sigma_\theta^2\right)p_{\mu\theta} - \rho\sigma_\theta^2\sigma_\eta^2\right\} \\
\gamma_1 &= \frac{\sigma_\mu^2}{\sigma_\zeta^2}\left\{3\rho\sigma_\zeta^2\sigma_\mu^2 p_{\mu\theta}^3 - 2\delta\rho\sigma_\eta^2\sigma_\zeta^2\sigma_\mu^2 p_{\mu\theta}^2 + \sigma_\eta^2\left(\sigma_\mu^2 + \rho\left(\sigma_\zeta^2 + \sigma_\mu^2\right)\right)p_{\mu\theta} - \delta\sigma_\eta^4\left(\sigma_\zeta^2 + \sigma_\mu^2\right)\right\} \\
\gamma_0 &= -\frac{\sigma_\theta^2}{\sigma_\zeta^2} p_{\mu\theta}\left(p_{\mu\theta} - \delta\sigma_\eta^2\right)\left(\sigma_\eta^2\sigma_\mu^2 + \sigma_\zeta^2(\sigma_\eta^2 + \sigma_\mu^2 p_{\mu\theta}^2)\right)
\end{aligned}
$$

That is, equation (5) is a cubic polynomial in $q_{\mu\theta}$, so it always has at least one real root.

(ii) (a) When $\lambda = 0$, $p_{\mu\theta} = 0$ so $\gamma_0 = 0$ and therefore one of the solutions of (6) is $q_{\mu\theta} = 0$.

(b) When $\lambda > 0$, $p_{\mu\theta} < 0$ and $\gamma_0, \gamma_1, \gamma_2, \gamma_3 < 0$. By Descartes' rule of signs the polynomial in (6) has no positive roots, but from above it has at least one real root. Therefore that root must be negative.

(iii) The financial market equilibrium is unique when equation (6) has a unique real root. This is true when the discriminant of the polynomial in (6), $\Delta^{q_{\mu\theta}}$, is non-positive. The discriminant is

$$
\Delta^{q_{\mu\theta}} = 18\gamma_3\gamma_2\gamma_1\gamma_0 - 4\gamma_2^3\gamma_0 + \gamma_2^2\gamma_1^2 - 4\gamma_3\gamma_1^3 - 27\gamma_3^2\gamma_0^2.
$$

□

Remark 2. A slightly stronger condition for existence and uniqueness is that

$$
3\gamma_3\gamma_1 - \gamma_2^2 \geq 0
$$

because

$$-27\gamma_3^2\Delta^{q_{\mu\theta}} = \left(2\gamma_2^3 - 9\gamma_3\gamma_2\gamma_1 + 27\gamma_3^2\gamma_0^2\right)^2 - 4\left(\gamma_2^2 - 3\gamma_3\gamma_1\right)^3.$$

Proof of Theorem 1. The value of information is

$$e^{\delta\psi_0(\lambda)} = \frac{\mathbb{E}\left[E_1\left(x_1^{u*};\mathcal{F}_1^u\right)\middle|\mathcal{F}_0^u\right]}{\mathbb{E}\left[\mathbb{E}\left[E_1\left(x_1^{i\,*};\mathcal{F}_1^i\right)\middle|\mathcal{F}_0^i\right]\middle|\mathcal{F}_0^u\right]} = \frac{\mathbb{E}\left[E_1\left(x_1^{u*};\mathcal{F}_1^u\right)\middle|\mathcal{F}_0^u\right]}{\mathbb{E}\left[E_1\left(x_1^{i\,*};\mathcal{F}_1^i\right)\middle|\mathcal{F}_0^u\right]},$$

where the last equality follows from the law of iterated expectations. I need to calculate two conditional expectations, which are very similar. For $j = i, u$, plugging $x_1^{j\,*}$ back into the value function gives

$$\begin{aligned} E_1\left(x_1^{j\,*};\mathcal{F}_1^j\right) = \sqrt{K_1^j}\exp\Bigg\{ &-\frac{1}{2\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)} \\ &\Big\{\mathbb{E}^2\left[P_2 - P_1|\mathcal{F}_1^j\right] - 2h^j\mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right]\mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right] \\ &+ \frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)}\mathbb{E}^2\left[D_3 - P_2|\mathcal{F}_1^j\right]\Big\}\Bigg\}. \end{aligned}$$

To calculate the conditional expectation of $E_1\left(x_1^{j\,*};\mathcal{F}_1^j\right)$, apply lemma 2 for $j = i, u$ with $X_1 = \mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right]$ and $X_2 = \mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right]$ conditionally on $\mathcal{F}_0^u$. Because ex-ante all random variables have zero means

$$\begin{aligned} \mu_1 &= \mathbb{E}\left[\mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right]\middle|\mathcal{F}_0^u\right] = 0 \\ \mu_2 &= \mathbb{E}\left[\mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right]\middle|\mathcal{F}_0^u\right] = 0. \end{aligned}$$

Moreover, take

$$\begin{aligned} \sigma_1^2 &= \mathrm{Var}\left(\mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right]\middle|\mathcal{F}_0^u\right) \\ \sigma_2^2 &= \mathrm{Var}\left(\mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right]\middle|\mathcal{F}_0^u\right) \\ \sigma_{12} &= \mathrm{Cov}\left(\mathbb{E}\left[P_2 - P_1|\mathcal{F}_1^j\right], \mathbb{E}\left[D_3 - P_2|\mathcal{F}_1^j\right]\middle|\mathcal{F}_0^u\right) \end{aligned}$$

and

$$\begin{aligned} b_1 &= b_2 = 0 \\ a_{11} &= -\frac{1}{2\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)} \\ a_{12} &= -h^j a_{11} \\ a_{22} &= \frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)}a_{11}. \end{aligned}$$

Then

$$|A| = a_{11}^2 \left[\frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)} - {h^j}^2 \right] = -\frac{a_{11}}{2\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)}$$

where the last equality follows from the variance decomposition in remark 1 and as a result

$$\begin{aligned}
S &= 1 - 2a_{11}\left[\sigma_1^2 - 2h^j\sigma_{12} + \frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)}\sigma_2^2\right] - 2a_{11}\frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)} \\
&= 1 + \frac{\sigma_1^2 - 2h^j\sigma_{12} + {h^j}^2\sigma_2^2}{\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)} + \frac{\left(\frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^j\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)} - {h^j}^2\right)\sigma_2^2}{\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)} \\
&\quad + \frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)} \\
&= \frac{\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_0^u\right)\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_0^u\right)}{\mathrm{Var}\left(P_2 - P_1 - h^j(D_3 - P_2)|\mathcal{F}_1^j\right)\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^j\right)}.
\end{aligned}$$

The last equality follows from remark 1, the definition of h^j and lemma 1. Now define K_0^{ju} to be the inverse of S for $j = i, u$ as in the last equation. Finally, the application of lemma 2 gives that the value of information is

$$\begin{aligned}
e^{\delta\psi_0(\lambda)} &= \frac{\mathbb{E}\left[E_1\left(x_1^{u*};\mathcal{F}_1^u\right)\middle|\mathcal{F}_0^u\right]}{\mathbb{E}\left[E_1\left({x_1^i}^*;\mathcal{F}_1^i\right)\middle|\mathcal{F}_0^u\right]} = \sqrt{\frac{K_1^u}{K_1^i}}\sqrt{\frac{K_0^{uu}}{K_0^{iu}}} \\
&= \frac{\mathrm{Var}(D_3 - P_2|\mathcal{F}_2^u)}{\mathrm{Var}(D_3 - P_2|\mathcal{F}_2^i)}\frac{\mathrm{Var}\left(P_2 - P_1 - h^u(D_3 - P_2)|\mathcal{F}_1^u\right)}{\mathrm{Var}\left(P_2 - P_1 - h^i(D_3 - P_2)|\mathcal{F}_1^i\right)}.
\end{aligned}$$

□

Proof of Corollary 3. The result follows by writing h^j in terms of β^j for $j = i, u$, and carrying out the algebra in the expression of theorem 1. □

Proof of Proposition 4. Lemma 3 yields that $h^i < h^u < 0$, thus $-\left(h^i\right)^2 < \left(h^u\right)^2 < 0$. Moreover, by lemma 1, $\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^u\right) > \mathrm{Var}\left(D_3 - P_2|\mathcal{F}_2^i\right)$, $\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^u\right) > \mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^i\right)$ and $\mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^u\right) > \mathrm{Var}\left(P_2 - P_1|\mathcal{F}_1^i\right)$. Remark 1 and theorem 1 now imply that $\psi_0(\lambda) > 0$ for any $\lambda \in [0, 1]$. □

Proof of Proposition 5. Combining the expressions for $\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)$ and $\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right)$ from the Kalman filter I get

$$\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right) = \sigma_\theta^2\left[1 - \frac{\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)}{\sigma_\mu^2}\right]$$

which shows that $\mathrm{Var}\left(\tilde{\theta}_1|\mathcal{F}_1^u\right)$ is decreasing in $\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)$. □

Proof of Proposition 6. Ignoring all terms of order λ^2 and higher in (6) gives

$$-\frac{\sigma_\mu^2}{\sigma_\zeta^2}\delta\sigma_\eta^4\left(\sigma_\zeta^2 + \sigma_\mu^2\right)q_{\mu\theta} + \frac{\sigma_\theta^2}{\sigma_\zeta^2}\delta\sigma_\eta^4\left(\sigma_\zeta^2 + \sigma_\mu^2\right)p_{\mu\theta} = 0,$$

the solution to which is $q_{\mu\theta} = \frac{\sigma_\theta^2}{\sigma_\mu^2} p_{\mu\theta}$. Using this expression, calculating the value of information in theorem 1 and evaluating its derivative at $\lambda = 0$ proves the result. The approximation improves as ρ decreases because, by inspection of (6), when ρ is smaller the contribution of higher-order terms is smaller. □

Proof of Theorem 2. The result follows by solving (6), calculating the value of information and its derivative and setting $\lambda = 0$ and $\rho = 0$. The algebraic manipulations are very complex and are thus omitted. □

Proof of Proposition 7. Both parts of the proposition follow from the solution to the Kalman filter above. □

Lemma 4 (Hedging of the Informed Agents)**.** *The hedging coefficient of the informed is*

$$\beta^i = -1$$

Proof.

$$\beta^i = \frac{\mathrm{Cov}\left(D_3 - P_2, P_2 - P_1 \middle| \mathcal{F}_1^i\right)}{\mathrm{Var}(P_2 - P_1 | \mathcal{F}_1^i)} = \frac{\mathrm{Cov}\left(D_3 - P_2 - P_1, P_2 - P_1 \middle| \mathcal{F}_1^i\right)}{\mathrm{Var}(P_2 - P_1 | \mathcal{F}_1^i)} = -1.$$

because conditional on $\mathcal{F}_1^i$, D_3 and $P_2 - P_1$ are independent by equation (9). □

Lemma 5 (Capital Gains Under Persistent Supply)**.** *Conditional on first-period information, when $\rho = 1$, capital gains from trade are independent of dividend information and supply.*

Proof. The result always holds for the informed. For the uninformed,

$$\begin{aligned}\mathrm{Cov}\left(P_2 - P_1, \tilde{\mu} \middle| \mathcal{F}_1^u\right) &= p_\mu^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) + p_\theta^C \mathrm{Cov}(\tilde{\mu}, \tilde{\theta}_1 | \mathcal{F}_1^u) \\ &= q_{\mu\theta} p_\theta^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) - q_{\mu\theta} p_\theta^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) = 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}\left(P_2 - P_1, \tilde{\theta}_1 \middle| \mathcal{F}_1^u\right) &= p_\mu^C \mathrm{Cov}(\tilde{\mu}, \tilde{\theta}_1 | \mathcal{F}_1^u) + p_\theta^C \mathrm{Var}(\tilde{\theta}_1 | \mathcal{F}_1^u) \\ &= -q_{\mu\theta}^2 p_\theta^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) + q_{\mu\theta}^2 p_\theta^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) = 0\end{aligned}$$

by equation (9) and corollary 2. □

Proof of Proposition 8. Lemma 5 establishes that $\mathrm{Var}(P_2 - P_1 | \mathcal{F}_1^u) = \mathrm{Var}(P_2 - P_1 | \mathcal{F}_1^i)$. Moreover,

$$\begin{aligned}&\mathrm{Cov}\left(D_3 - P_2, P_2 - P_1 \middle| \mathcal{F}_1^u\right) \\ &= -p_\eta^2 \sigma_\eta^2 + p_\mu^D p_\mu^C \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) + \left(p_\mu^D p_\theta^C + p_\mu^C p_\theta^D\right) \mathrm{Cov}(\tilde{\mu}, \tilde{\theta}_1 | \mathcal{F}_1^u) + p_\theta^D p_\theta^C \mathrm{Var}(\tilde{\theta}_1 | \mathcal{F}_1^u) \\ &= -p_\eta^2 \sigma_\eta^2 + \left[p_\mu^D p_\mu^C - q_{\mu\theta}\left(p_\mu^D p_\theta^C + p_\mu^C p_\theta^D\right) - q_{\mu\theta}^2 p_\theta^D p_\theta^C\right] \mathrm{Var}(\tilde{\mu} | \mathcal{F}_1^u) \\ &= -p_\eta^2 \sigma_\eta^2 + 0\end{aligned}$$

by equation (8), equation (9) and corollary 2. This establishes that

$$\mathrm{Cov}\left(D_3 - P_2, P_2 - P_1 \middle| \mathcal{F}_1^u\right) = -\mathrm{Var}(P_2 - P_1 | \mathcal{F}_1^u).$$

Therefore $\beta^u = -1$. The expression for the value of information now follows by corollary 3. In particular, the value of information is

$$e^{2\delta\psi_0} = \frac{\mathrm{Var}\left(D_3 | \mathcal{F}_2^u\right)}{\mathrm{Var}\left(D_3 | \mathcal{F}_2^i\right)} \frac{\mathrm{Var}\left(D_3 | \mathcal{F}_1^u\right)}{\mathrm{Var}\left(D_3 | \mathcal{F}_1^i\right)} \frac{\mathrm{Var}\left(D_3 - P_2 | \mathcal{F}_1^i\right)}{\mathrm{Var}\left(D_3 - P_2 | \mathcal{F}_1^u\right)}$$

Consider the first two fractions in this expression. The denominators are constant, because

$$\mathrm{Var}\left(D_3|\mathcal{F}_2^i\right) = \sigma_\zeta^2 = \mathrm{Var}\left(D_3|\mathcal{F}_1^i\right)$$

The numerators are equal,

$$\mathrm{Var}\left(D_3|\mathcal{F}_2^u\right) = \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_2^u\right) + \sigma_\zeta^2 = \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right) + \sigma_\zeta^2 = \mathrm{Var}\left(D_3|\mathcal{F}_1^u\right)$$

by corollary 2 and proposition 7. The last term is

$$\frac{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^i\right)}{\mathrm{Var}\left(D_3 - P_2|\mathcal{F}_1^u\right)} = \frac{\sigma_\zeta^2 + p_\eta^2\sigma_\eta^2}{\sigma_\zeta^2 + \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right) + p_\eta^2\sigma_\eta^2}$$

where, in addition, $p_\eta = p_\theta$ by equation (8) and corollary 2. The value of information is

$$e^{2\delta\psi_0} = \left(\frac{\sigma_\zeta^2 + \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)}{\sigma_\zeta^2}\right)^2 \frac{\sigma_\zeta^2 + p_\theta^2\sigma_\eta^2}{\sigma_\zeta^2 + \mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right) + p_\theta^2\sigma_\eta^2}$$

where p_θ and $\mathrm{Var}\left(\tilde{\mu}|\mathcal{F}_1^u\right)$ are functions of λ. Taking the derivative of $e^{2\delta\psi_0}$ with respect to λ and carrying out the algebra shows that the value of information is decreasing in λ. □

B Continuous-Time Appendix

B.1 The Filtering Problem of the Uninformed Agents

The matrices in the solution of the filtering problem of the uninformed agents are

$$q_{xx} = \begin{bmatrix} p_\theta^2\sigma_\theta^2 + p_\mu^2\sigma_\mu^2 & 0 \\ 0 & \sigma_D^2 \end{bmatrix},$$

and

$$h = \frac{1}{H}\begin{bmatrix} \frac{1}{p_\theta}\frac{p_\mu}{p_\theta}\sigma_\mu^2(\phi_\theta + G) & \sigma_\theta^2\frac{\sigma_\mu^2}{\sigma_D^2}\phi_D \\ \frac{1}{p_\theta}\sigma_\theta^2(\phi_\mu + G) & -\frac{p_\mu}{p_\theta}\sigma_\theta^2\frac{\sigma_\mu^2}{\sigma_D^2}\phi_D \end{bmatrix},$$

where

$$H = \left(\frac{p_\mu}{p_\theta}\right)^2\sigma_\mu^2(\phi_\theta + G) + \sigma_\theta^2\left(\phi_\mu + G\right)$$

and

$$G = \sqrt{\frac{\phi_\theta^2\left(\frac{p_\mu}{p_\theta}\right)^2\sigma_\mu^2 + \sigma_\theta^2\left(\phi_\mu^2 + \frac{\sigma_\mu^2\phi_D^2}{\sigma_D^2}\right)}{\sigma_\theta^2 + \left(\frac{p_\mu}{p_\theta}\right)^2\sigma_\mu^2}}.$$

B.2 Portfolio Choice

For $j = i, u$, the value function is

$$J^j(W_t^j, S_t^j) = -e^{-r\delta W_t^j - \frac{1}{2}\left(\alpha^j + S_t^{jT}\gamma^j S_t^j\right)}.$$

α^j is the scalar

$$\alpha^j = \frac{1}{r}\mathrm{tr}\left(\gamma^j v_S^j v_S^{jT}\right) + 2\left[\frac{\nu}{r} + \ln(r) - 1\right],$$

where γ^u is a (2×2) matrix and γ^i is a (3×3) matrix. In particular, for $j = i, u$, γ^j is the solution to the Algebraic Riccati Equation

$$\begin{aligned} 0 = \gamma \left[v_R^j v_R^{jT} \left(m_S^j - \frac{r}{2} I \right) - v_S^j v_R^{jT} m_R^j \right] + \left[v_R^j v_R^{jT} \left(m_S^{jT} - \frac{r}{2} I \right) - m_R^{jT} v_R^j v_S^{jT} \right] \gamma \\ + \gamma v_S^j \left(v_R^{jT} v_R^j - v_R^j v_R^{jT} I \right) v_S^{jT} \gamma + m_R^{jT} m_R^j \end{aligned}$$

The optimal demand coefficient for $j = i, u$ is the matrix

$$d^j = \frac{1}{r\delta v_R^j v_R^{jT}} \left(m_R^j - v_R^j v_S^{jT} \gamma^j \right).$$

For uninformed agents,

$$\begin{aligned} m_R^u &= \left[\begin{array}{cc} (p_\mu + p_{\hat{\mu}})\phi_\mu m - rp_0 & -(r + \phi_\theta)(p_\theta + p_{\hat{\theta}}) \end{array} \right], \\ v_R^u &= \left(\left[\begin{array}{cc} 0 & p_D \end{array} \right] + \left[\begin{array}{cc} (p_\mu + p_{\hat{\mu}}) & (p_\theta + p_{\hat{\theta}}) \end{array} \right] h \right) (q_{xx})^{\frac{1}{2}}, \\ m_S^u &= \left[\begin{array}{cc} 0 & 0 \\ 0 & -\phi_\theta \end{array} \right], \\ v_S^u &= \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] h (q_{xx})^{\frac{1}{2}}. \end{aligned}$$

For informed agents,

$$\begin{aligned} m_R^i &= \left[\begin{array}{ccc} (p_\mu + p_{\hat{\mu}})\phi_\mu m - rp_0 & -r(p_\theta + p_{\hat{\theta}}) & -r \left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta} \right) \end{array} \right] \\ &\quad + \left[\begin{array}{ccc} 0 & (p_\theta + p_{\hat{\theta}}) & \left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta} \right) \end{array} \right] m_S^i, \\ v_R^i &= \left[\begin{array}{ccc} 0 & p_D \sigma_D & 0 \end{array} \right] + (p_\mu + p_{\hat{\mu}}) \left[\begin{array}{ccc} 0 & 0 & \sigma_\mu \end{array} \right] \\ &\quad + \left[\begin{array}{ccc} 0 & (p_\theta + p_{\hat{\theta}}) & \left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta} \right) \end{array} \right] v_S^i, \\ m_S^i &= \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -\phi_\theta & 0 \\ 0 & 0 & -\phi_e \end{array} \right], \\ v_S^i &= \left[\begin{array}{ccc} 0 & 0 & 0 \\ \sigma_\theta & 0 & 0 \\ \frac{1}{H} \frac{p_\mu}{p_\theta} \sigma_\mu^2 \sigma_\theta (\phi_\theta + G) & \frac{1}{H} \sigma_\theta^2 \frac{\sigma_\mu^2 \phi_D}{\sigma_D} & -\frac{1}{H} \sigma_\theta^2 \sigma_\mu (\phi_\mu + G) \end{array} \right], \end{aligned}$$

where

$$\phi_e = \frac{1}{H} \left\{ \left(\frac{p_\mu}{p_\theta} \right)^2 \sigma_\mu^2 \phi_\theta (\phi_\theta + G) + \sigma_\theta^2 \left[\frac{\sigma_\mu^2 \phi_D^2}{\sigma_D^2} + \phi_\mu (\phi_\mu + G) \right] \right\}.$$