

Optimal Taxation of Wealthy Individuals:

Online Appendix

Ali Shourideh

Wharton School, University of Pennsylvania

shouride@wharton.upenn.edu

December 28, 2012

Contents

1	Introduction	2
2	Generalizations	2
3	Private Productivities and Bequest Taxes	5
4	Incomplete Market Model	8

1 Introduction

This is the online (or Supplementary) appendix for the paper “Optimal Taxation of Wealthy Individuals”. It contains various related results not included in either the paper or its appendix. Included are:

1. Generalization of the progressivity of (outside) saving wedge to general utility functions and distribution of shocks.
2. Discussion of the results of the infinite horizon model when θ_t is private.
3. An ad-hoc incomplete market model a la [Angeletos \(2007\)](#) and a formula for characterizing the tail of the wealth distribution.

2 Generalizations

In this section, I discuss the extent to which results in section 2 of the paper generalize for a more general class of utility functions and distribution functions. I argue that some of the results discussed above can be extended to a more general environment.

Taxes on Outside Saving. I start by considering taxes on outside savings. To establish ideas, I first consider general distribution of shocks for ε and log-preferences when θ is observable. To proceed we make the following assumptions:

Assumption 1 *The shock ε is distributed according to $H(\varepsilon)$ with p.d.f. function $h(\varepsilon)$ that satisfies the following properties:*

1. Support of ε is $\mathbb{R}_+$, i.e., $h(\varepsilon) > 0, \forall \varepsilon \in \mathbb{R}_+$,
2. The function $\rho(\varepsilon) = -1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)}$ is increasing in ε ,
3. $\lim_{\varepsilon \rightarrow 0} -1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} = -\underline{x}$ exists and is finite.

The above assumptions are standard in the moral hazard literature: the first implies that the mechanism designer cannot learn anything the household’s action from a single observation; the second implies that the likelihood ratio $\frac{g_k}{g}$ is increasing in y ; the third is required for existence of the solution (see [Mirrlees \(1999\)](#)).¹ Under these assumptions and with log-utility we can show a similar result to theorem 1 in [Shourideh \(2012\)](#):

¹Given that $-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)}$ is an increasing function of ε and that $\int \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)}\right) h(\varepsilon) d\varepsilon = 0$, $\underline{x}$ must be positive.

Proposition 1 Suppose Assumption 1 holds. Further, suppose that

$$q\bar{\theta} - 1 \leq \rho \left[\min_{x \in [0, \frac{1}{x}]} \frac{1}{x} + \beta \int \log \left(1 + x \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right) \right) \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right) dH(\varepsilon) \right]^{-1}$$

Then in the optimal allocation with public θ , saving wedge $\tau_S(\theta)$, as defined in equation (3) in Shourideh (2012), is an increasing function of θ .

The proof is identical to the case with gamma distribution and is available upon request. The intuition behind this result is as before. The planner would like more productive types to invest more but that requires higher degree of risk in their consumption in the second period and hence their demand for risk-free assets increases.² This intuition leads to a progressive tax on outside saving. Note that the case illustrated in Shourideh (2012) is more convenient and can be interpreted easily since consumption in the second period is linear in ε . This does not hold true for a general distribution.

With general utility function, average consumption is not equated across types. However, one can show that in general, $E_\theta \frac{1}{u'(c_1)}$ is equated across types. This makes the interpretation of results about taxes on the inside saving rather hard. However, it can be shown that under a fairly general set of assumptions, saving wedge remains progressive. We have the following analogue to Theorem 1 in Shourideh (2012):

Proposition 2 Suppose that utility function $u(c)$ is given by $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$ and that assumption 1 holds. Then there exists $\hat{\theta}(\sigma, h(\cdot)) > 0$ such that if $\bar{\theta} < \hat{\theta}(\sigma, h(\cdot))$, then in the optimal allocation with public θ , saving wedge $\tau_S(\theta)$ is an increasing function of θ .

Proof. Note that in this case, the first order conditions are given by the following:

$$\begin{aligned} c_0^{-\sigma} - \lambda_0 + \rho\sigma\zeta k_1 c_0^{-\sigma-1} &= 0 \\ \beta c_1^{-\sigma} - \lambda_1 + \beta\zeta c_1^{-\sigma} \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right) &= 0 \\ \lambda_1 \theta - \lambda_0 - \rho\zeta c_0^{-\sigma} &= 0 \end{aligned}$$

²As before, average consumption in the second period is equated across types, i.e., $E[c_1(\theta, \varepsilon) | \theta] = E[c_1(\theta', \varepsilon) | \theta']$.

After much simplification and similar to the log-case

$$\begin{aligned}\beta \int \frac{1}{1-\sigma} \left(\beta \lambda_1 + \beta \zeta \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right) \right)^{\frac{1-\sigma}{\sigma}} (-h(\varepsilon) - \varepsilon h'(\varepsilon)) d\varepsilon &= \rho k_1 c_0^{-\sigma} \\ \lambda_1 \theta - \lambda_0 - \rho \zeta c_0^{-\sigma} &= 0 \\ c_0^{-\sigma} - \lambda_0 + \rho \sigma \zeta k_1 c_0^{-\sigma-1} &= 0\end{aligned}$$

These three equations determine ζ , k_1 and c_0 . Let

$$\Phi(\zeta) = \beta \int \frac{1}{1-\sigma} \left(\beta \lambda_1 + \beta \zeta \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right) \right)^{\frac{1-\sigma}{\sigma}} (-h(\varepsilon) - \varepsilon h'(\varepsilon)) d\varepsilon$$

Note that

$$\begin{aligned}c_0 &= \rho (\lambda_1 \theta - \lambda_0)^{-\frac{1}{\sigma}} \zeta^{\frac{1}{\sigma}} \\ \rho k_1 c_0^{-\sigma} &= \Phi(\zeta).\end{aligned}$$

So

$$(\lambda_1 \theta - \lambda_0) \zeta^{-1} - \lambda_0 + \rho \sigma \zeta (\lambda_1 \theta - \lambda_0)^{\frac{1}{\sigma}} \zeta^{-\frac{1}{\sigma}} \Phi(\zeta) = 0$$

Now the rest of the argument is similar to the one in Theorem 1 in [Shourideh \(2012\)](#). Note that the above function is a U-shaped function that tends to ∞ as ζ converges to 0. Furthermore, it is increasing in θ and for θ low enough has at most two solution. Since the lower solution is less risky, the objective is higher for the lower value of ζ . Then it is clear that this solution must be increasing in θ – this is because the above function is a decreasing function of ζ at the lower solution. This proves that $\zeta(\theta)$ is increasing in θ . The rest of the proof is identical to the proof of Theorem 1 in [Shourideh \(2012\)](#). ■

Taxes on Inside Saving. Note that when we depart from the setting discussed in the body of the paper, log-preferences and gamma distribution, taxes on inside saving are not linear any more. Recall that inside saving wedge, $\tau_K(\theta)$, is defined as

$$\frac{1}{1 - \tau_K(\theta)} = \beta \frac{1}{u'(c_0(\theta))} \int_0^\infty u'(c_1(\theta, \varepsilon)) \varepsilon \theta dH(\varepsilon)$$

A similar analysis as in section 2.3 in [Shourideh \(2012\)](#) can be performed. It can be shown that

$$\frac{1}{u'(c_1(\theta, \varepsilon))} = \gamma(\theta) + \zeta(\theta) \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)} \right)$$

for some $\gamma(\theta)$ and $\zeta(\theta)$. Assuming that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, then we can write

$$\frac{1}{1 - \hat{\tau}_K(\theta)} = \beta c_0(\theta)^\sigma \theta \int_0^\infty \frac{\varepsilon h(\varepsilon)}{\gamma(\theta) + \zeta(\theta) \left(-1 - \frac{\varepsilon h'(\varepsilon)}{h(\varepsilon)}\right)} d\varepsilon$$

Under the assumption that $\varepsilon \sim \Gamma\left(\eta, \frac{1}{\eta}\right)$, we can rewrite the above as

$$\frac{1}{1 - \hat{\tau}_K(\theta)} = \beta \frac{c_0(\theta)^\sigma \theta}{\eta \zeta(\theta)} \int_0^\infty \log(\gamma(\theta) + \zeta(\theta) \eta(\varepsilon - 1)) (-h(\varepsilon) - \varepsilon h'(\varepsilon)) d\varepsilon$$

Note that in the case of log preferences, the above coincides with $\frac{\theta k(\theta)}{\zeta(\theta)}$. As in the log case, there are forces for and against progressively of the (inside) saving wedge. Consumption in the first period, $c_0(\theta)^\sigma$, θ , and the last integral are increasing in θ , while $\frac{1}{\zeta(\theta)}$ is decreasing in θ ; a similar result to the log case.

3 Private Productivities and Bequest Taxes

In this section, I consider the dynamic extension when θ_t is privately known to the households. I assume that incentive constraints can be replaced with local ones. This implies that the component planning problem would be as in section 3 except that the following constraint is added:

$$U'(\theta) = \frac{1}{\theta} \frac{u'(c_0(\theta))}{k(\theta)} \quad (1)$$

Note that the component planning problem is still homogeneous in w and hence, Proposition 2 holds. In what follows I discuss the implications of the model with private θ_t for the timing of consumption over time as well as optimal bequest taxes.

As before, one can show that the modified inverse Euler equation is satisfied and is unchanged. The intuition is as before: when young, saving tightens incentive constraints while it relaxes future incentive constraints when old. In order to prove that bequests should be subsidized we need to assume that upon deviation about type, θ_t , consumption increases. This property, although cannot be proven analytically, is true in all numerical simulations. We, hence, have the following proposition:

Proposition 3 Suppose that in the solution to the component planning problem above

$$\left. \frac{\partial}{\partial \hat{\theta}} c_0(\hat{\theta}, w) + k_1(\hat{\theta}, w) \left(1 - \frac{\hat{\theta}}{\theta}\right) \right|_{\hat{\theta}=\theta} > 0$$

and that $\mu(\theta) < 0$, where $\mu(\theta)$ is the multiplier associated with (1). Then

$$\frac{\beta}{q_{t+1}} E_{t+1} u'(c_{0,t+2}) < u'(c_{1,t}).$$

That is bequests should be subsidized.

Proof. Suppose that $\left. \frac{\partial}{\partial \hat{\theta}} c_0(\hat{\theta}, w) + k(\hat{\theta}) \left(1 - \frac{\hat{\theta}}{\theta}\right) \right|_{\hat{\theta}=\theta} > 0$ and that $\mu(\theta) < 0$. The first condition implies that

$$c'_0(\theta) > \frac{1}{\theta} k(\theta)$$

where we have suppressed dependences of consumption and investment on w . Since $\mu(\theta) < 0$, we must have

$$-\mu(\theta) c'_0(\theta) > -\mu(\theta) \frac{1}{\theta} k(\theta)$$

Note that the FOC with respect to c_0 is given by

$$-1 + \gamma(\theta) u'(c_0(\theta)) - \left(\frac{1}{\theta} \mu(\theta) + \hat{\xi}(\theta) \right) k(\theta) u''(c_0(\theta)) = 0$$

where $\gamma(\theta) = \lambda - \frac{1}{f(\theta)} (\mu(\theta) f(\theta))'$. Hence

$$-f(\theta) + [\lambda f(\theta) - (\mu(\theta) f(\theta))'] u'(c_0(\theta)) - \frac{1}{\theta} k(\theta) \mu(\theta) f(\theta) u''(c_0(\theta)) - \hat{\xi}(\theta) f(\theta) u''(c_0(\theta)) = 0$$

Integrating the above and rearranging gives

$$\begin{aligned} \lambda \int u'(c_0(\theta)) dF(\theta) &= 1 + \int_{\Theta} \left[(\mu(\theta) f(\theta))' u'(c_0(\theta)) + \frac{1}{\theta} k(\theta) \mu(\theta) f(\theta) u''(c_0(\theta)) \right] d\theta \\ &\quad + \int_{\Theta} \hat{\xi}(\theta) u''(c_0(\theta)) dF(\theta) \\ &< 1 + \int_{\Theta} [(\mu(\theta) f(\theta))' u'(c_0(\theta)) + \mu(\theta) f(\theta) c'_0(\theta) u''(c_0(\theta))] d\theta \\ &\quad + \int_{\Theta} \hat{\xi}(\theta) u''(c_0(\theta)) dF(\theta) \\ &< 1 + \int_{\Theta} [(\mu(\theta) f(\theta))' u'(c_0(\theta)) + \mu(\theta) f(\theta) c'_0(\theta) u''(c_0(\theta))] d\theta \\ &= 1 + \int_{\Theta} d(\mu(\theta) f(\theta) u'(c_0(\theta))) = 1 + \mu(\theta) f(\theta) u'(c_0(\theta)) \Big|_{\underline{\theta}}^{\bar{\theta}} = (2) \end{aligned}$$

where the first inequality follows the above assumption and the second inequality follows from the fact that $\hat{\zeta}(\theta) > 0$; see lemma 4 in the paper. Now, given (2), the claim can be proved the same way as in Theorem 2. ■

4 Incomplete Market Model

In this section, I provide a formula for the tail of the stationary distribution of wealth in an incomplete market model without taxes. The incomplete market version of the model is very similar to [Angeletos \(2007\)](#). For simplicity, we assume that $u(c) = \log c$. A similar analysis holds for general CRRA utility functions. Suppose that when young, households have two options for investment: invest in the risky project with production function $(\varepsilon_{t+1}\theta_t k_{t+1})^\alpha l_{t+1}^{1-\alpha}$, or to borrow and lend using a risk free bond. When old, households hire labor to produce output form a competitive labor market with wage p_t . When old, the households leave bequest for their descendants. Given this market structure, the budget constraints for the young and old households are given by

$$\begin{aligned} c_{0,t} + k_{t+1} + b_{t+1} &= R_t a_t + p_t \\ c_{1,t} + a_{t+2} &= (\varepsilon_{t+1}\theta_t k_{t+1})^\alpha l_{t+1}^{1-\alpha} - p_{t+1} l_{t+1} + R_{t+1} b_{t+1} \end{aligned}$$

An equilibrium is defined as the solution to the following problem

$$V_t(a) = \max \int \left[\log c_0 + \beta \log c_1 + \beta^2 V_{t+2}(a') \right] dGdF \quad (\text{P}') \quad (1)$$

subject to the budget constraints above as well as

$$a_t \geq -h_t = - \sum_{j=0}^{\infty} \frac{p_{t+2j}}{R_t \cdots R_{t+2j}}.$$

together with market clearing. Note that the above borrowing constraint is a natural debt limit and h_t is the present value of labor income by future generations and it can be interpreted as human capital. Further R_t and p_t are determined so that

$$\begin{aligned} \int b_{t+1}(a, \theta_t) dF(\theta_t) d\psi_{0,t}(a) + \int a d\psi_{1,t-1}(a) &= 0 \\ \int l_{t+1}(a, \theta_t, \varepsilon_{t+1}) dH(\varepsilon_{t+1}) dF(\theta_t) d\psi_{1,t}(a) &= 1 \end{aligned}$$

where $\psi_{0,t}$ and $\psi_{1,t-1}$ are the distributions of asset for the young and the old at period t . As before, profit maximization implies that

$$(1 - \alpha) (\varepsilon_{t+1}\theta_t k_{t+1})^\alpha l_{t+1}^{-\alpha} = p_{t+1}$$

and hence

$$\begin{aligned}\pi_t &= (\varepsilon_{t+1}\theta_t k_{t+1})^\alpha l_{t+1}^{1-\alpha} - p_{t+1}l_{t+1} = \alpha \left(\frac{1-\alpha}{p_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \varepsilon_{t+1}\theta_t k_{t+1} \\ &= \hat{\kappa}_t \varepsilon_{t+1} \theta_t k_{t+1}\end{aligned}$$

Once labor demand is determined, the above problem is a classic portfolio problem studied by [Samuelson \(1969\)](#). The utility function is homothetic while the budget set is linear in allocations. This means that the policy functions are linear in an appropriate state variable. Because of the existence of labor income, assets are not the state variable. However, we can show that if we define

$$\begin{aligned}\hat{a}_t &= a_t + h_t \\ \hat{b}_{t+1} &= b_t + \frac{h_{t+2}}{R_{t+1}}\end{aligned}$$

Then the budget constraints become

$$\begin{aligned}c_{0,t} + k_{t+1} + \hat{b}_{t+1} &= \hat{a}_t R_t \\ c_{1,t} + \hat{a}_{t+2} &= \pi_t + R_{t+1} \hat{b}_{t+1}\end{aligned}$$

where $\hat{a}_t$ and $\hat{b}_t$ are physical asset together with present value of future generations labor income – what can be interpreted as human capital. Given this definition, we have the following theorem³:

Proposition 4 *The policy functions in (P'), satisfy the following*

$$\begin{aligned}k_t(\theta, a) &= s_{k,t}(\theta, R_{t+1}) \beta R_t (a + h_t) \\ b_{t+1}(\theta, a) &= s_{b,t}(\theta, R_{t+1}) \beta R_t (a + h_t) - \frac{h_{t+2}}{R_{t+1}} \\ c_{0,t}(\theta, a) &= (1 - \beta) R_t (a + h_t) \\ c_{1,t}(\varepsilon, \theta, a) &= (1 - \beta) \beta (\hat{\kappa}_{t+1} \varepsilon \theta s_{k,t}(\theta, R_{t+1}) + R_{t+1} s_{b,t}(\theta, R_{t+1})) R_t (a + h_t) \\ a_{t+2}(\varepsilon, \theta, a) &= \beta^2 (\hat{\kappa}_{t+1} \varepsilon \theta s_{k,t}(\theta, R_{t+1}) + R_{t+1} s_{b,t}(\theta, R_{t+1})) R_t (a + h_t) - h_t\end{aligned}$$

where $s_{k,t}(\theta, R_{t+1}) + s_{b,t}(\theta, R_{t+1}) = 1$ and

$$\int_0^\infty \frac{\hat{\kappa}_{t+1} \varepsilon \theta - R_{t+1}}{s_{k,t}(\theta, R_{t+1}) (\hat{\kappa}_{t+1} \varepsilon \theta - R_{t+1}) + R_{t+1}} dH(\varepsilon) = 0 \quad (3)$$

³The analysis here closely follows that of [Angeletos \(2007\)](#).

The above result is familiar from [Samuelson \(1969\)](#) as well more recently [Angeletos \(2007\)](#). With log utility, the total saving rate is β . Furthermore, the break-down between bond and equity is given by the portfolio choice equation (3). In the appendix, we show that $s_{k,t}(\theta, R)$ is increasing in θ .

In the above model, steady state implies that

$$\beta^2 \int (\hat{\kappa} \varepsilon s_k(\theta, R) + R(1 - s_k(\theta, R))) dF(\theta) dH(\varepsilon) = 1$$

and hence from Jensen's inequality

$$\int \log \left[\beta^2 (\hat{\kappa} \varepsilon s_k(\theta, R) + R(1 - s_k(\theta, R))) \right] dF(\theta) dH(\varepsilon) < 0$$

This implies that $a_t + h_t$ converges almost surely to zero. That is since households can borrow against their descendants labor income, over time, they accumulate debt so that their financial wealth, i.e., bequest, is negative and equal to the negative of their human capital.

When instead of the above borrowing constraint, we impose a constraint of the form $a' \geq \underline{a}$ where $\underline{a} > -h_t$, then a similar analysis as in the paper shows that the new policy functions, $a'_c(a, \theta, \varepsilon)$, satisfy

$$\lim_{a \rightarrow \infty} \frac{\partial}{\partial a} a'_c(a, \theta, \varepsilon) = \beta^2 (\hat{\kappa} \varepsilon s_k(\theta, R) + R s_b(\theta, R)) R$$

where R is the stationary interest rate, $\hat{\kappa}$ is derived given stationary wages and s_k and s_b are defined as in Proposition 4. Hence, using [Mirek \(2011\)](#)'s result, we can show that the stationary distribution for wealth has a Pareto tail and the formula for this tail ratio, ν , is given by

$$\beta^{2\nu} \int_{\Theta \times \mathbb{R}_+} (\hat{\kappa} \varepsilon s_k(\theta, R) + R(1 - s_k(\theta, R)))^\nu dF(\theta) dH(\varepsilon) = 1$$

References

- ANGELETOS, G. (2007): "Uninsured Idiosyncratic Investment Risk and Aggregate Saving," *Review of Economic Dynamics*, 10, 1–30. [2](#), [8](#), [9](#), [10](#)
- MIREK, M. (2011): "Heavy Tail Phenomenon and Convergence to Stable Laws for Iterated Lipschitz Maps," *Probability Theory and Related Fields*, 151, 705–734. [10](#)

- MIRRLEES, J. (1999): "The Theory of Moral Hazard and Unobservable Behaviour: Part I," *Review of Economic Studies*, 66, 3–21. 2
- SAMUELSON, P. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *The Review of Economics and Statistics*, 51, 239–246. 9, 10
- SHOURIDEH, A. (2012): "Optimal Taxation of Wealthy Individuals," Working Paper, University of Pennsylvania. 2, 3, 4