

# Dynamic Managerial Compensation: On the Optimality of Seniority-based Schemes\*

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## Abstract

We study the optimal dynamics of incentives for a manager whose ability to generate cash flows changes stochastically with time and is his private information. We show that, in general, the power of incentives (or "pay for performance") may either increase or decrease with tenure. However, risk aversion and high persistence of ability call for a reduction in the power of incentives later in the relationship. Our results follow from a new variational approach that permits us to tackle directly the "full program", thus bypassing many of the difficulties of working with the "relaxed program" encountered in the literature.

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*Keywords:* managerial compensation, power of incentives, pay for performance, dynamic mechanism design, adverse selection, moral hazard, persistent productivity shocks, risk aversion.

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# 1 Introduction

In dynamic business environments, the ability of top managers to generate profits for their firms is expected to change with time as a result, for example, of changes in the organization, the arrival of new technologies, or market consolidations. A key difficulty is that, while such changes are largely expected, their implications for profitability typically remain the managers' private information. In this paper we ask the following questions: Should managers be induced to work harder at the beginning of their employment relationships or later on? Should the variation be stronger for managers of low or of high initial productivity? And, how should "pay for performance" change over the course of the employment relationship to sustain the desired dynamics of effort?

We consider an environment where, at the time of joining the firm, the manager possesses private information about his productivity (i.e., his ability to generate cash flows). This private information originates, for instance, in tasks performed in previous contractual relationships, as well as in personal traits that are not directly observable by the firm. The purpose of the analysis is to examine the implications of such evolving private information for the dynamic provision of incentives.

In the environment described above, a firm finds it expensive to ask a manager to exert more effort for three reasons. First, higher effort is costly for the manager and must be compensated. Second, asking higher effort of a manager with a given productivity requires increasing the compensation promised to all managers with higher productivity. This compensation is required even if the firm does not ask the more productive managers to exert more effort and represents an additional "rent" for these managers. It is needed to discourage them from mimicking the less productive managers by misrepresenting their productivity and reducing their effort. Third, inducing higher effort requires pay to be more sensitive to performance. This, in turn, exposes the manager to more volatility in his compensation. When the manager is risk averse, this increase in volatility reduces his expected payoff, requiring higher compensation by the firm.

The above effects of effort on compensation shape the way the firm induces its managers to respond to productivity shocks over time. First, consider the case where managers are expected to be risk neutral.<sup>1</sup> The concern for reducing the rent left to those managers whose initial productivity is high leads the firm to distort downward (relative to the first best) the level of effort asked of those managers whose initial productivity is low. We focus on the case where the effect of the initial productivity on future productivity is expected to decline with time, as this property seems reasonable in most cases of interest. Under this assumption, the firm optimally asks higher effort later in the relationship.<sup>2</sup> Interestingly, effort increases most over time for those managers whose

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<sup>1</sup>The reason why, in this case, the firm cannot extract all the surplus by "selling out" the business to the manager is that, as explained above, the manager typically possesses private information about his ability to generate cash flows and hence about his value for the business.

<sup>2</sup>This property of declining distortions is quite common in the literature on dynamic contracting with adverse

initial productivity is lowest, since the initial distortions in effort for such managers are largest.

Next consider the case where the managers are expected to be risk averse. Mitigating the volatility of compensation calls for reducing the power of incentives later in the relationship. The reason is that, viewed from the date the contract is initially agreed, managers face greater uncertainty about their productivity at later dates. Whether the rent effect or the risk effect prevail as the length of the employment relationship grows (i.e., whether incentives become stronger or weaker with time) then depends on factors such as the manager's initial productivity, the degree of productivity persistence, and the manager's degree of risk aversion.

In this paper we develop a simple, yet flexible, framework that permits us to investigate the implications of the above trade-offs both for dynamics of effort and the power of incentives under optimal contracts. When the manager is risk averse, the relationship between effort and pay for performance need not be straightforward. One reason is that the manager's responsiveness to incentives depends also on the *level* of his pay. A risk-averse manager who expects to be paid a lot, for instance due to good past performance, is less willing to exert high effort for any given sensitivity of pay to performance. This is simply because he values additional compensation less. A related difficulty is that a manager may be rewarded for high cash flows both through contemporaneous and future payments. Future payments can depend in turn on the manager's future actions and productivity realizations. This potentially blurs the relationship between pay for performance and effort.

To avoid these complications, we suggest a new definition of the "*power of incentives*". In any given period, it is the ratio between the manager's marginal disutility of effort and the marginal utility of his compensation in that period, evaluated at their equilibrium levels. Our rationale for this definition is that, when this ratio is high, the manager's rewards for generating additional cash flows must also be high. This is either because the manager's marginal disutility of effort is high, or because his marginal utility of additional pay is low, so that he is difficult to motivate. If one considers compensation schemes which are differentiable in the firm's cash flows and which depend only on the cash flows generated in the period of compensation, then incentive compatibility requires the derivative be equal to the power of incentives. In this sense, the power of incentives measures the sensitivity of payments to cash flows "locally". Our definition can therefore be applied to compensation schemes which are non-linear as a function of the firm's performance, a necessary generalization given that linear schemes need not be optimal when the manager is risk averse. It also applies to compensation schemes that are non-differentiable in cash flows, such as those implemented with a combination of fixed pay, stocks, and options.<sup>3</sup>

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selection (see, e.g., Baron and Besanko, 1984, Besanko, 1985, and Battaglini, 2005, among others). One of the key contributions of the current paper is to study the extent to which such results are robust to the possibility that the agent is risk averse.

<sup>3</sup>This flexibility is valuable given that we cannot exclude the possibility that, for certain specifications, the optimal

It turns out that the dynamics of the power of incentives under optimal contracts are more easily understood than the dynamics of effort choices. Indeed the characterization of optimal effort policies, except for special cases, has been notoriously difficult.<sup>4</sup> Our analysis uses novel variational arguments to tackle directly the firm’s “full problem”. That is, we directly account for all of the manager’s incentive constraints. For any incentive-compatible contract, we identify certain “admissible variations”, by which we mean perturbations to the contract which *preserve* incentive compatibility. For a contract to be optimal, these perturbations must not increase the firm’s profits. This requirement implies a new set of Euler equations that jointly determine how the ex-ante expectation of the power of incentives evolves with the manager’s tenure in the firm. The admissible variations we identify do not always permit us to characterize how effort responds to all possible contingencies. However, they do permit fairly robust predictions as to how, on average, effort and the power of incentives evolve over time under fully optimal contracts.

Our approach permits us to bypass many of the difficulties encountered in the literature. The typical approach involves solving for the optimal contract while imposing only a restricted set of incentive constraints, usually referred to as “local” constraints. In other words, one first solves a “relaxed problem”. One then seeks to identify restrictions on the primitive environment that guarantee that the solution to the relaxed problem satisfies the remaining incentive constraints. When the agent is risk averse, explicit solutions to the relaxed problem can be obtained only numerically. Furthermore, checking the remaining incentive constraints can typically be done only on a case-by-case basis.<sup>5</sup> On the other hand, when validated, the relaxed approach has the advantage of yielding predictions about the dynamics of the power of incentives that depend on the *realized productivity history* rather than simply on the *average* over histories.

While our approach of tackling directly the full program yields predictions that hold only on average, it has the advantage that the predictions are robust to different primitive specifications. Further, predictions that hold only on average seem important for empirical work, given that histories of productivity shocks are unlikely to be unobservable by the econometrician.<sup>6</sup>

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dynamics of effort may be sustained only with non-differentiable schemes.

<sup>4</sup>In our model, characterizing optimal effort policies is difficult whenever the manager is risk averse. This difficulty is the same one observed by Edmans and Gabaix (2011), who argue that “the full contracting problem is usually intractable as there is a continuum of possible effort choices”. Edmans and Gabaix therefore focus on a particular environment, where a careful balancing of the costs and benefits of additional effort is unnecessary. In their setting, optimal effort is constant over time and equal to the highest feasible level.

<sup>5</sup>The relaxed approach fails whenever the effort policies that solve the relaxed problem fail to satisfy certain “monotonicity conditions” necessary for incentive compatibility (for the present paper, see Lemma 3 in the Appendix). We refer the reader to Pavan, Segal, and Toikka (2012) and Battaglini and Lamba (2012) for further discussion of how the relaxed approach may fail in quasilinear settings.

<sup>6</sup>The existing empirical literature has taken a non-structural approach to analyzing how pay-to-performance changes with tenure. For this literature, qualitative predictions about how the power of incentives changes on average may be of greatest interest.

**Implications for empirical work.** Our results contribute to the debate about how managerial incentives ought to change over a manager’s tenure, and what explains the observed patterns. The empirical literature often focuses on a measure of “pay for performance” proposed by Jensen and Murphy (1990). This is the responsiveness of CEO pay to changes in shareholder wealth. When applied to compensation schemes which depend only on current-period performance (mentioned above), our definition of the power of incentives mirrors the measures typically used in this literature.

The evidence of how pay-for-performance varies with managerial tenure is mixed. More recent work finds that the sensitivity of pay to performance increases with tenure. Gibbons and Murphy (1992), Lippert and Porter (1997), and Cremers and Palia (2010) support this view, while Murphy (1986) and Hill and Phan (1991) find evidence of the opposite. A number of theories have been proposed to explain these patterns. Gibbons and Murphy (1992) provide a model of career concerns to suggest that explicit pay-for-performance ought to increase closer to a manager’s retirement. Edmans et al. (2012) suggest a similar conclusion but based on the idea that, with fewer remaining periods ahead, replacing current pay with future promised utility becomes more difficult to sustain. Arguments for the opposite finding have often centered on the possibility that managers capture the board once their tenure has grown large (see, e.g., Hill and Phan (1991) and Bebchuk and Fried (2004)), while Murphy (1986) proposes a theory based on market learning about managerial quality over time.

Our paper contributes to this debate by indicating that the key determinant for whether the power of incentives ought to increase or decrease with tenure may be the manager’s *degree of risk aversion*. Another prediction of our model, although one which is subject to the limitations of the relaxed approach discussed above, is that the increase in the power of incentives over time is most pronounced (equivalently, the decrease is smaller) for those managers whose initial productivity is low. Because productivity is positively correlated with performance, this result suggests a negative correlation between early performance and the increase in the power of incentives over the course of the employment relationship. This prediction seems a distinctive feature of our theory, albeit one that, to the best of our knowledge, has not been tested yet.

**Organization of the paper.** The rest of the paper is organized as follows. We briefly review some pertinent literature in the next section. Section 3 then describes the model while Section 4 characterizes the firm’s optimal contract. Section 5 concludes. All proofs are in the Appendix at the end of the manuscript.

## 2 Related literature

The literature on managerial compensation is obviously too vast to be discussed within the context of this paper. We refer the reader to Prendergast (1999) for an excellent overview and to Edmans

and Gabaix (2009) for a survey of some recent developments. Below, we limit our discussion to the papers that are most closely related to our own work.

Our work is related to the literature on “dynamic moral hazard” and its application to managerial compensation. Seminal works in this literature include Lambert (1983), Rogerson (1985), and Spear and Srivastava (1987). These works provide qualitative insights about optimal contracts but do not provide a full characterization. This has been possible only in restricted settings: Phelan and Townsend (1991) characterize optimal contracts numerically in a discrete-time model, while Sannikov (2008) characterizes the optimal contract in a continuous-time setting with Brownian shocks.<sup>7</sup> In contrast to these works, Holmstrom and Milgrom (1987) show that the optimal contract has a simple structure when (a) the agent does not value the timing of payments, (b) noise follows a Brownian motion and (c) the agent’s utility is exponential and defined over consumption net of the disutility of effort. Under these assumptions, the optimal contract takes the form of a simple linear aggregator of aggregate profits.

Contrary to the above works, in the current paper we assume that, in each period, the manager observes the shock to his productivity before choosing effort.<sup>8</sup> In this respect, our paper is closely related to Laffont and Tirole (1986) who first proposed this alternative timing. This timing permits one to use techniques from the mechanism design literature to solve for the optimal contract. The same approach has been recently applied to dynamic managerial compensation by Edmans and Gabaix (2011) and Edmans et al. (2012). Our model is similar in spirit, but with a few key distinctions. First, we assume that the manager is privately informed about his initial productivity before signing the contract; this is what drives the result that the managers must be given a strictly positive share of the surplus. A second key difference is that we characterize how effort and the power of incentives in the optimal contract evolve over time.<sup>9</sup>

Our paper is also related to our previous work on managerial turnover in a changing world (Garrett and Pavan, 2012). In that paper we assumed that all managers are risk neutral and focused on the dynamics of retention decisions. In contrast, in the present paper, we abstract from retention (i.e., assume a single manager) and focus instead on the effect of risk aversion on the dynamics of the power of incentives and effort.

A growing number of papers study optimal financial instruments in dynamic principal-agent relationships. For instance, DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), Sannikov

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<sup>7</sup>See also Sadzik and Stacchetti (2013) for recent work on the relationship between discrete-time and continuous-time models.

<sup>8</sup>We abstract from the possibility that performance is affected by transitory noise that occurs after the manager chooses his effort. It is often the case, however, that compensation can be structured so that it continues to implement the desired effort policies even when performance is affected by transitory noise.

<sup>9</sup>As mentioned in footnote 4, the above work assumes that it is optimal to induce the highest feasible effort constantly over time.

(2007),<sup>10</sup> and Biais et al. (2010) study optimal financial contracts for a manager who privately observes the dynamics of cash flows and can divert funds from investors to private consumption. In these papers, it is typically optimal to induce the highest possible effort (which is equivalent to no stealing/no saving); the instrument which is then used to create incentives is the probability of terminating the project. One of the key findings is that the optimal contract can often be implemented using long-term debt, a credit line, and equity. The equity component represents a linear component to the compensation scheme which is used to make the agent indifferent as to whether or not to divert funds to private use. Since the agent’s cost of diverting funds is constant over time and output realizations, so is the equity share. In contrast, we provide an explanation for why and how this share may change over time. While these papers suppose that cash-flows are i.i.d., Tchisty (2006) explores the consequences of correlation and shows that the optimal contract can be implemented using a credit line with an interest rate that increases with the balance. As in Tchisty (2006), we also assume that managerial productivity is imperfectly correlated over time.<sup>11</sup>

From a methodological standpoint, we draw from recent results in the dynamic mechanism design literature. In particular, the new approach we follow here builds on the techniques recently developed in Pavan, Segal, and Toikka (2012). That paper provides a general treatment of incentive compatibility in dynamic settings. It extends previous work by Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007), and Kapicka (forthcoming), among others, by allowing for more general payoffs and stochastic processes and by identifying the role of impulse responses as the key driving force for the dynamics of optimal contracts.

Relative to the aforementioned body of work on dynamic mechanism design, the key methodological contribution of the present paper is in the way we arrive to optimal contracts. As explained above, this involves identifying perturbations of the proposed policies that preserve incentive compatibility and then using variational arguments to identify the key properties. To the best of our knowledge, this approach is completely new to all of the papers in the dynamic mechanism design literature. One of its key advantages, as noted above, is that it permits us to identify fairly robust properties of optimal contracts for *risk-averse* agents.<sup>12</sup> In this respect, the paper is also related to the literature on optimal dynamic taxation (also known as Mirrleesian taxation, or new

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<sup>10</sup>As in our work, and contrary to the other papers cited here, Sannikov (2007) allows the agent to possess private information prior to signing the contract. Assuming the agent’s initial type can be either “bad” or “good”, he characterizes the optimal separating menu where only good types are funded.

<sup>11</sup>Other recent papers that consider persistent private information are He (2008), Edmans, Gabaix, Sadzik, and Sannikov (2011), Strulovici (2012), and Williams (2011). Contrary to these papers and the current one, DeMarzo and Sannikov (2008), Bergemann and Hege (1998, 2005), and Horner and Samuelson (2012), consider an environment in which both the investors and the agent learn about the firm’s true productivity over time and where the agent’s beliefs about the likely success of the project differ from the investors’ only in case the agent deviates, in which case the divergence in beliefs may be persistent.

<sup>12</sup>For static models with risk aversion, see Salanie (1990), and Laffont and Rochet (1998).

public finance). Recent contributions to this literature include Battaglini and Coate (2008), Zhang (2009), Golosov, Troshkin, and Tsyvinski (2012) and Farhi and Werning (2013). A complication encountered in this literature is that, because of risk aversion, policies solving the relaxed program can only be computed numerically; likewise, the incentive-compatibility of such policies can only be checked with numerical methods. The approach introduced in the present paper may perhaps prove useful for characterizing certain properties of optimal dynamic taxes (as well as optimal contracts for risk-averse agents in other settings) by allowing one to bypass this difficulty.

### 3 The Model

#### 3.1 The environment

**Players, actions, and information.** The firm's shareholders (hereafter referred to as the principal) hire a manager to work on a project for two periods. In each period  $t = 1, 2$ , the manager receives some private information  $\theta_t \in \Theta_t = [\underline{\theta}_t, \bar{\theta}_t]$  about his ability to generate cash flows for the firm (his type). After observing  $\theta_t$ , he then chooses effort  $e_t \in E = \mathbb{R}$ . The latter, combined with the manager's productivity  $\theta_t$ , then leads to cash flows  $\pi_t$  according to the simple technology  $\pi_t = \theta_t + e_t$ .

Both  $\theta \equiv (\theta_1, \theta_2)$  and  $e \equiv (e_1, e_2)$  are the manager's private information. Instead, the cash flows  $\pi \equiv (\pi_1, \pi_2)$  are verifiable, and hence can be used as a basis for the manager's compensation.

**Payoffs.** For simplicity, we assume no discounting.<sup>13</sup> The principal's payoff is the sum of the firm's cash flows in the two periods, net of the manager's compensation, i.e.

$$U^P(\pi, c) = \pi_1 + \pi_2 - c_1 - c_2,$$

where  $c_t$  is the period- $t$  compensation to the manager and where  $c \equiv (c_1, c_2)$ . The function  $U^P$  is also the principal's Bernoulli utility function used to evaluate possible lotteries over  $(\pi, c)$ .

By choosing effort  $e_t$  in period  $t$ , the manager suffers a disutility  $\psi(e_t)$ . The manager's Bernoulli utility function is then given by

$$U^A(c, e) = v(c_1) + v(c_2) - \psi(e_1) - \psi(e_2), \tag{1}$$

where  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing, weakly concave, surjective, Lipschitz continuous, and differentiable function. The case where  $v$  is linear corresponds to the case where the manager is risk neutral, while the case where  $v$  is strictly concave corresponds to the case where he is risk averse. Note that the above payoff specification also implies that the manager has preferences for consumption smoothing. This assumption is common in the dynamic moral hazard (and taxation) literature

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<sup>13</sup>None of the results hinge on this assumption.

(a few notable exceptions are Holmstrom and Milgrom (1987) and more recently Edmans and Gabaix (2011)).<sup>14</sup> We denote the inverse of the felicity function by  $w$  (i.e.,  $w \equiv v^{-1}$ ).

**Productivity process.** The manager’s first-period productivity,  $\theta_1$ , is drawn from an absolutely continuous c.d.f.  $F_1$  with density  $f_1$  strictly positive over  $\Theta_1$ . His second-period productivity is drawn from an absolutely continuous c.d.f.  $F_2(\cdot|\theta_1)$  with density  $f_2(\cdot|\theta_1)$  strictly positive over a subset  $\Theta_2(\theta_1) = [\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$  of  $\Theta_2$ .

While our results extend to more general processes, to ease the exposition we will focus on the case where  $\theta_t$  follows an autoregressive process so that  $\tilde{\theta}_2 = \gamma\tilde{\theta}_1 + \tilde{\varepsilon}$ , with  $\tilde{\varepsilon}$  drawn from a continuously differentiable c.d.f.  $G$  with support  $[\underline{\varepsilon}, \bar{\varepsilon}]$ .<sup>15</sup> We assume that  $\gamma \geq 0$ , so that higher period-1 productivity leads to higher period-2 productivity in the sense of first-order stochastic dominance. We will refer to  $\gamma = 1$  as to the case of “fully persistent productivity” (meaning that, holding effort fixed, the effect of any shock to period-1 productivity on the firm’s average cash flows is constant over time). We will be primarily interested in the case where  $\gamma \in [0, 1]$ .

**Effort disutility.** As is standard in the literature (see, e.g., Laffont and Tirole (1986)), we assume that there exists an arbitrarily large threshold  $\bar{e} > 0$  such that  $\psi$  is thrice continuously differentiable over  $(0, \bar{e})$  with  $\psi'(e), \psi''(e) > 0$  and  $\psi'''(e) \geq 0$  for all  $e \in (0, \bar{e})$ , and that  $\psi'(e) > 1$  for all  $e > \bar{e}$ . We then further restrict  $\psi$  to be quadratic over the relevant range  $[0, \bar{e}]$  and linear elsewhere. More precisely, we assume that  $\psi(e) = 0$  for all  $e \leq 0$ ,  $\psi(e) = e^2/2$  for all  $e \in [0, \bar{e}]$ , and  $\psi(e) = \bar{e}e - \bar{e}^2/2$  for all  $e > \bar{e}$ . That the disutility is quadratic over  $[0, \bar{e}]$  permits us to identify a convenient family of perturbations to incentive-compatible contracts that preserve incentive compatibility. The reason for assuming that  $\psi$  is linear above  $\bar{e}$  is that this ensures that  $\psi$  is Lipschitz continuous. This last property in turn guarantees that both the efficient and the profit-maximizing effort levels are interior, while also ensuring that the manager’s payoff under any incentive-compatible contract is equi-Lipschitz continuous in his productivity and hence can be conveniently expressed through a differentiable envelope formula (as explained below). We suppose throughout that  $\bar{e}$  can be chosen large enough so that the principal never asks for effort which exceeds this level.

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<sup>14</sup>As is standard, this specification presumes that the manager’s period- $t$  consumption  $c_t$  coincides with the period- $t$  compensation. In other words, it abstracts from the possibility of secret private saving. The specification also presumes time consistency. This means that, in both periods, the manager maximizes the expectation of  $U^A$ , where the expectation clearly depends on all available information.

<sup>15</sup>Results qualitatively similar to those in Propositions 2 and 3 below obtain under more general processes satisfying the following two properties: (i) the kernels can be ranked according to first-order stochastic dominance (FOSD) and (ii) on average, the impulse responses of future productivity to initial productivity decline with time. Formally, let  $\theta_2 = z(\theta_1, \varepsilon)$ , where  $\varepsilon$  is a shock independent of  $\theta_1$ . Then let the impulse response of  $\theta_2$  to  $\theta_1$  be the derivative of  $z$  with respect to  $\theta_1$ . In the case of a linear autoregressive process  $\theta_2 = z(\theta_1, \varepsilon) = \gamma\theta_1 + \varepsilon$ , so that the impulse response is equal to the persistence parameter  $\gamma$ . The assumption of FOSD is then equivalent to requiring that  $z$  is nondecreasing in  $\theta_1$  while the assumption that impulse responses decline, on average, with time is equivalent to requiring that the average derivative of  $z$  with respect to  $\theta_1$  is less than one, where the average is over all possible histories.

### 3.2 The principal's problem

The principal's problem is to choose a contract specifying for each period a recommended effort choice along with a compensation that conditions on the observed cash flows. It is convenient to think of such a contract as a mechanism  $\Omega \equiv \langle \xi, x \rangle$  comprising a recommended *effort policy*  $\xi \equiv (\xi_1(\theta_1), \xi_2(\theta))$  and a *compensation scheme*  $x \equiv (x_1(\theta_1, \pi_1), x_2(\theta, \pi))$ .

The effort that the firm recommends in period one is naturally restricted to depend on the manager's self-reported productivities  $\theta = (\theta_1, \theta_2)$  only through  $\theta_1$ . This property reflects the assumption that the manager learns his period-2 productivity only at the beginning of the second period, as explained in more detail below. The effort that the firm recommends in the second period does not depend on the first-period cash flow. This can be shown to be without loss of optimality for the firm, a consequence of the simplifying assumptions that (i) cash flows are deterministic functions of effort and productivity, and (ii) the manager is not protected by limited liability. The compensation paid in each period naturally depends both on the reported productivities and the observed cash flows.<sup>16</sup>

Importantly, we assume that the firm offers the manager the contract after he is already informed about his initial productivity  $\theta_1 \in \Theta_1$ . After receiving the contract, the manager then chooses whether or not to accept it. If he rejects it, he obtains an outside continuation payoff which we assume to be equal to zero for all possible types. If, instead, he accepts it, he is then bound to stay in the relationships for the two periods.<sup>17</sup> He is then asked to report his productivity  $\hat{\theta}_1 \in \Theta_1$  and his recommended effort  $\xi_1(\hat{\theta}_1)$ . The manager then privately chooses effort  $e_1$  which combines with the manager's productivity  $\theta_1$  to give rise to the period-1 cash flows  $\pi_1 = \theta_1 + e_1$ . After observing the cash flows  $\pi_1$ , the firm then pays the manager a compensation  $x_1(\hat{\theta}_1, \pi_1)$ .

The functioning of the contract in period two parallels the one in period one. At the beginning of the period, the manager learns his new productivity  $\theta_2$ . He then updates the principal by sending a new report  $\hat{\theta}_2 \in \Theta_2$ . The contract then recommends effort  $\xi_2(\hat{\theta})$  which may depend on the entire

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<sup>16</sup>By reporting variations in his productivity, the manager effectively adjusts his compensation package in response to new information. This seems consistent with the practice of managers proposing changes to their compensation, which has become quite common (see, among others, Bebchuk and Fried, 2004, and Kuhnen and Zwiebel, 2008). However, note that the allocations sustained under the optimal contract as determined below will typically be sustainable also without the need for direct communication between the manager and the firm (this is true, in particular, when there is a one-to-one mapping from the equilibrium cash flows to the manager's productivity).

<sup>17</sup>We expect that our results are not significantly affected if we instead require period-2 individual rationality, provided that the manager's outside option when leaving the firm is not too large (when the manager's outside option is sufficiently small, the period-2 individual rationality constraint is slack, so the solution to the firm's problem is precisely the one we characterize below). One reason that the manager's outside option at period two may be small is that the manager anticipates adverse treatment by the labor market in case he leaves the firm early. Fee and Hadlock (2004), for instance, document evidence for a labor market penalty in case a senior executive leaves the firm early, although the size of this penalty appears to depend on the circumstances surrounding departure.

history  $\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2)$  of reported productivities. The manager then privately chooses effort  $e_2$  which, together with  $\theta_2$ , leads to the cash flows  $\pi_2$ . After observing  $\pi_2$ , the firm then pays the manager a compensation  $x_2(\hat{\theta}, \pi)$  and the relationship is terminated.

As usual, we will restrict attention to contracts that are accepted by all types and that induce the manager to report truthfully and follow the principal's recommendations in each period.<sup>18</sup> We will refer to such contracts as *individually rational* and *incentive compatible*.

## 4 Profit-maximizing Contracts

### 4.1 Representation of the firm's profits

We begin by deriving an expression for the firm's profits under any individually-rational and incentive-compatible contract that will prove instrumental for the characterization of the optimal policies. When the manager is risk neutral (when  $v$  is linear), the firm's profits can be expressed entirely in terms of the effort policy  $\xi$ . When, instead, the manager is risk averse, the firm's profits will depend also on the first-period compensation; the second-period compensation is then determined by incentive compatibility according to a function of the effort policy and the period-one compensation described below.

We start by representing the manager's equilibrium payoff under any individually-rational and incentive-compatible contract  $\Omega$ . Let  $\pi_t(\theta) = \theta_t + \xi_t(\theta)$  and  $c_t(\theta) = x_t(\theta, \pi(\theta))$  denote, respectively, the firm's period- $t$  equilibrium cash flows and compensation to a manager whose productivity history is  $\theta$  (by "equilibrium", we mean under a truthful and obedient strategy for the manager).<sup>19</sup> Let  $V(\theta) = v(c_1(\theta_1)) + v(c_2(\theta)) - \psi(\xi_1(\theta_1)) - \psi(\xi_2(\theta))$  denote the manager's equilibrium payoff in state  $\theta$ . We then have the following result.

**Lemma 1** *Suppose that the contract  $\Omega = \langle \xi, x \rangle$  is individually rational and incentive compatible. The compensation that the firm provides in equilibrium to each manager whose lifetime productivity is  $\theta = (\theta_1, \theta_2)$  must satisfy<sup>20</sup>*

$$v(c_1(\theta_1)) + v(c_2(\theta)) = W(\theta; \xi) + K, \text{ where} \quad (2)$$

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<sup>18</sup>Note that the manager's second-period payoff does not depend directly on his first-period productivity. Hence, the environment is "Markov". This means that restricting attention to contracts that induce the manager to follow a truthful and obedient strategy in period two also after having departed from truthful and obedient behavior in period one is without loss of optimality.

<sup>19</sup>As explained above,  $\xi_1$  and  $x_1$  depend on  $(\theta, \pi)$  only through the period-1 observations. We abuse notation here to ease the exposition.

<sup>20</sup>To be precise, Condition (2) must hold with probability one; that is, it can be violated but only over a set of zero-probability measure.

$$\begin{aligned}
W(\theta; \xi) \equiv & \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta)) + \int_{\underline{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \\
& + \int_{\underline{\theta}_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ \int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\theta_1, s)) ds \right],
\end{aligned} \tag{3}$$

and where  $K$  is a nonnegative scalar.

The result in Lemma 1 is obtained by combining period-2 necessary conditions for incentive compatibility (as derived, for example, in Laffont and Tirole (1986)) with period-1 necessary conditions for incentive compatibility (as derived, for example, in Pavan, Segal and Toikka (2012); see also Garrett and Pavan (2012) for a similar application to a model of managerial turnover).

The result in the lemma highlights how the surplus that the firm must leave to each manager depends on the effort policy  $\xi$ . Consider a manager of initial productivity  $\theta_1$ . His expected payoff under any individually-rational and incentive-compatible contract must satisfy

$$\mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] = \mathbb{E}^{\tilde{\theta}|\underline{\theta}_1} [V(\tilde{\theta})] + \int_{\underline{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds. \tag{4}$$

The surplus that a type  $\theta_1$  expects above the one expected by the lowest period-1 type  $\underline{\theta}_1$  is thus increasing in the effort that the firm asks of managers with initial productivities  $\theta'_1 \in (\underline{\theta}_1, \theta_1)$  in each of the two periods. This surplus is necessary to dissuade type  $\theta_1$  from mimicking the behavior of these lower types. Such mimicry would involve, say, reporting a lower type in the first period and then replicating the distribution of that type's productivity reports in the second period. By replicating the same cash flows expected from a lower type, a higher type can obtain the same compensation while working less, thus economizing on the disutility of effort.

Consider the effect of asking higher effort of a low period-1 type on the surplus that the firm must leave to those managers whose initial productivity is higher. If productivity is only partially persistent (in our autoregressive model, if  $\gamma < 1$ ), then asking for this increase in effort at period two has a smaller effect than asking for it in period one. The reason is that the amount of effort that the higher period-1 type expects to be able to save relative to the lower period-1 type he mimics is smaller in the second period, reflecting that the initial productivity is imperfectly persistent. As we will see below, this property plays an important role in shaping the dynamics of effort and the power of incentives under optimal contracts.

Now note that the scalar  $K$  in Lemma 1 corresponds to the expected payoff  $\mathbb{E}^{\tilde{\theta}|\underline{\theta}_1} [V(\tilde{\theta})]$  of the lowest period-1 type. Using (4), it is easy to see that, if the lowest period-1 type finds it optimal to accept the contract (i.e., if  $K \geq 0$ ), then so does any manager whose initial productivity is higher. Furthermore, it is easy to see that, fixing the effort policy  $\xi$ , the firm's profits are always maximized by setting  $K = 0$ , as we do throughout the rest of the paper. The above observations then permit us to express the firm's expected profits as a function of the effort policy  $\xi$  and the period-1 equilibrium compensation  $c_1$  as follows.

**Lemma 2** Consider any individually-rational and incentive-compatible contract  $\Omega \equiv \langle \xi, x \rangle$  yielding an expected surplus of zero to a manager with the lowest period-1 productivity  $\underline{\theta}_1$ . The firm's expected profit from  $\Omega$  is given by

$$\mathbb{E}[U^P] = \mathbb{E} \left[ \tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - c_1(\tilde{\theta}_1) - w \left( W(\tilde{\theta}; \xi) - v(c_1(\tilde{\theta}_1)) \right) \right], \quad (5)$$

where  $c_1(\theta_1) = x_1(\theta_1, \pi_1(\theta_1))$  is the compensation paid in equilibrium to a manager whose period-1 productivity is  $\theta_1$ , and where  $\pi_1(\theta_1) = \theta_1 + \xi_1(\theta_1)$  is the equilibrium cash flow generated by the same manager.

Note that, when the manager is risk neutral ( $v(y) = w(y)$  for all  $y$ ), the result in Lemma 2 implies that the firm's payoff is equal to the entire surplus of the relationship, net of a term that corresponds to the surplus that must be left to the manager and which depends only on the effort policy  $\xi$ :

$$\mathbb{E}[U^P] = \mathbb{E} \left[ \begin{array}{l} \tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) - \psi(\xi_1(\tilde{\theta}_1)) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - \psi(\xi_2(\tilde{\theta})) \\ - \int_{\underline{\theta}_1}^{\tilde{\theta}_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \end{array} \right]. \quad (6)$$

The expression in (6) is what in the dynamic mechanism design literature is referred to as “dynamic virtual surplus”.

As one should expect, when instead the manager is risk averse, the firm's payoff depends not only on the effort policy but also on the way the compensation is spread over the two periods. The value of the result in Lemma 2 comes from the fact that the choice over such compensation can be reduced to the choice over the period-1 compensation. The result in the lemma uses the property that any two contracts implementing the same effort policy  $\xi$  must give the manager the same utility of compensation not just in expectation, but with probability one (i.e., except over at most a set of productivity histories  $\theta = (\theta_1, \theta_2)$  of zero measure). This equivalence result (which is the dynamic analog in our non-quasilinear environment of the celebrated “revenue equivalence” for static quasilinear problems) plays an important role below in the characterization of the optimal policies.<sup>21</sup>

## 4.2 Optimal policies

We now consider the question of which contracts maximize the firm's expected profits. Note that the firm's profits depend on the contract  $\Omega \equiv \langle \xi, x \rangle$  only through (i) the effort policy  $\xi$  and (ii) the equilibrium period-1 compensation  $c_1$ , as defined in Lemma 2. Given  $\xi$  and  $c_1$ , the period-2 equilibrium compensation  $c_2(\theta) = x_2(\theta, \pi(\theta))$  is then uniquely determined by the need to provide the manager a lifetime utility of monetary compensation equal to the level required by incentive

<sup>21</sup>See Pavan, Segal, and Toikka (2012) for a more general analysis of payoff-equivalence in dynamic settings.

compatibility, as given by (2). That is,  $c_2(\theta) = w(W(\theta; \xi) - v(c_1(\theta_1)))$ . On the other hand, the payment scheme  $x \equiv (x_1(\theta_1, \pi_1), x_2(\theta, \pi))$  is not uniquely determined for values of the cash flows inconsistent with truthful and obedient behavior by the manager; that is, for  $\pi_t \neq \pi_t(\theta) \equiv \theta_t + \xi_t(\theta)$ . Below, we propose a definition of the power of incentives and develop an approach that permits us to dispense with the need to specify how the payment scheme responds "off-path" to such cash flows.

As noted in the Introduction, the approach typically followed in the dynamic mechanism design literature to identify the optimal policies is the following. First, consider a *relaxed program* that replaces all relevant incentive-compatibility constraints with condition (2) of Lemma 1; this permits one to express the principal's payoff as "dynamic virtual surplus". In our environment, this means choosing policies  $(\xi_1, \xi_2, c_1)$  to solve the unconstrained maximization of the firm's profits in (5). Condition (2) of Lemma 1 is necessary but not sufficient for incentive compatibility. Therefore, one must typically identify auxiliary assumptions on the primitives of the problem guaranteeing that the solution to the relaxed program can indeed be sustained in a contract that is truly individually rational and incentive compatible for the manager. Identifying such primitive conditions is not always simple, but it is often possible when the manager is risk neutral. It can be difficult when the manager is risk averse.<sup>22</sup>

The approach we follow here is therefore different. Because the firm's profits under any individually-rational and incentive-compatible contract must be consistent with the representation in (5), we use this expression to evaluate the performance of different contracts. However, because not all policies  $(\xi_1, \xi_2, c_1)$  are implementable under an incentive-compatible contract (in particular, this may be the case for those policies that maximize (5)), we do not aim at maximizing this expression directly. Instead, we use simple variational arguments to identify the optimal policies. More precisely, we first identify "admissible variations". By this we mean perturbations which, when applied to policies that can be implemented under a contract that is individually rational and incentive compatible for the manager, guarantee that the new perturbed policies remain implementable under a contract with the same properties. For the candidate policies to be sustained under an optimal contract, it then must be the case that no admissible variation increases the firm's profits, as expressed in (5).

The hurdle is thus to identify perturbations that preserve the manager's incentives. We show in Lemma 3 in the Appendix that the following conditions are necessary and sufficient for an effort

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<sup>22</sup>The additional difficulty relates to the complexity of characterizing the solutions to the relaxed program, which are the policies solving the Euler equations in Proposition 4 below. One can, however, proceed numerically on a case-by-case basis. This is the approach taken, for instance, in Farhi and Werning (2012), who encounter the same difficulties in their study of optimal dynamic taxation for risk-averse agents.

policy  $\xi$  to be implementable by an incentive-compatible contract. First, for all  $\theta_1, \hat{\theta}_1 \in \Theta_1$ ,

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left( \xi_1(\hat{\theta}_1) - (s - \hat{\theta}_1) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi' \left( \xi_2(\hat{\theta}_1, \tilde{\theta}_2) \right) \right] \right\} ds \\ & \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left( \xi_1(s) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi' \left( \xi_2(s, \tilde{\theta}_2) \right) \right] \right\} ds. \end{aligned}$$

Second, for all  $\theta_1$ ,  $\pi_2(\theta_1, \theta_2) \equiv \xi_2(\theta_1, \theta_2) + \theta_2$  is nondecreasing in  $\theta_2$ . The second requirement is the familiar monotonicity constraint from static mechanism design (e.g., Laffont and Tirole (1986)). The first condition is an “integral monotonicity condition” analogous to the one in Theorem 3 of Pavan, Segal and Toikka (2012), adapted to our managerial compensation setting. The key methodological contribution of the present paper is in showing how one can work directly with such a formula to identify perturbations that preserve incentive compatibility and then use them to arrive at properties of optimal contracts. Given our simplifying assumption that the disutility of effort is quadratic over the relevant range, these perturbations include translations of effort in either period by a uniform constant. In particular, for any implementable effort and consumption policy, one can obtain other implementable policies by translating effort and then adjusting the compensation so that the net payoffs continue to satisfy the conditions in Lemma 1 while maintaining  $\mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] = 0$ . The requirement that such perturbations do not increase profits then yields the following result.<sup>23</sup>

**Proposition 1** *Effort and compensation policies  $(\xi_1^*, \xi_2^*, c_1^*)$  sustained under any optimal contract must satisfy the following conditions:*

$$\mathbb{E} \left[ \psi' \left( \xi_1^*(\tilde{\theta}_1) \right) w' \left( v \left( c_1^*(\tilde{\theta}_1) \right) \right) \right] = 1 - \mathbb{E} \left[ \frac{\psi'' \left( \xi_1^*(\tilde{\theta}_1) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left( v \left( c_1^*(r) \right) \right) f_1(r) dr \right] \quad (7)$$

$$\begin{aligned} \mathbb{E} \left[ \psi' \left( \xi_2^*(\tilde{\theta}) \right) w' \left( v \left( c_2^*(\tilde{\theta}) \right) \right) \right] &= 1 - \gamma \mathbb{E} \left[ \frac{\psi'' \left( \xi_2^*(\tilde{\theta}) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left( v \left( c_1^*(r) \right) \right) f_1(r) dr \right] \\ &- \mathbb{E} \left[ \frac{\psi'' \left( \xi_2^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left( v \left( c_2^*(\tilde{\theta}_1, r) \right) \right) - w' \left( v \left( c_1^*(\tilde{\theta}_1) \right) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right] \end{aligned} \quad (8)$$

$$w' \left( v \left( c_1^*(\theta_1) \right) \right) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ w' \left( v \left( c_2^*(\theta_1, \tilde{\theta}_2) \right) \right) \right] \quad (9)$$

and

$$c_2^*(\theta) = w \left( W(\theta; \xi) - v \left( c_1^*(\theta_1) \right) \right)$$

<sup>23</sup>Note that such perturbations also preserve incentive compatibility in environments with more than two periods and richer stochastic processes. Euler conditions analogous to those in Proposition 1 can thus be obtained also for richer environments.

where  $W(\theta; \xi)$  is the total utility of compensation, as given by (3). The effort policy implemented under an optimal contract is essentially unique.<sup>24</sup> If  $v$  is strictly concave, the compensation policy implemented under an optimal contract is also essentially unique.

Condition (9) is the well-known inverse Euler equation

$$\frac{1}{v'(c_1^*(\theta_1))} = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ \frac{1}{v'(c_2^*(\theta_1, \tilde{\theta}_2))} \right]$$

first suggested by Rogerson (1985). It is obtained by considering variations in the period-1 compensation coupled with adjustments to the period-2 compensation that guarantee that the total utility that the manager derives from his life-time compensation continues to satisfy Condition (2). The key contribution is in deriving Conditions (7) and (8), which are new to the literature and obtained by considering the IC-preserving perturbations of the effort policy  $\xi$  described above. Contrary to the perturbations of the compensation scheme that lead to Condition (9), these perturbations necessarily change the manager's expected payoff, as one can readily see from (4).

### 4.3 Dynamics of the power of incentives

Our next objective is to understand what Conditions (7) and (8) in Proposition 1 imply for how the power of incentives optimally changes with tenure. First, we need a workable definition of the “power of incentives”.

**Definition 1 (Power of incentives)** For each  $t = 1, 2$  and each  $\theta$ , the (local) power of incentives under the (incentive-compatible) mechanism  $\langle \xi, x \rangle$  is defined to be the ratio

$$\frac{\psi'(\xi_t(\theta))}{v'(c_t(\theta))} = \psi'(\xi_t(\theta)) w'(v(c_t(\theta)))$$

between the marginal disutility of effort and the marginal utility of consumption, evaluated at  $c_t(\theta) = x_t(\theta, \pi(\theta))$ .

The rationale for this definition comes from the fact that, when this ratio is high, either because the marginal disutility of effort is high or because the marginal utility of consumption is low, the firm must resort to a high sensitivity of pay to performance to induce the desired level of effort.

To see this more clearly, consider payment schemes  $x$  where the payments in each period depend on the history of observed cash flows only through the contemporaneous observations (that is, for all  $t = 1, 2$ ,  $x_t(\theta, \pi)$  depends on  $\pi$  only through  $\pi_t$ ) and where each payment  $x_t(\theta, \pi)$  is differentiable

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<sup>24</sup>By essentially unique, we mean except over a zero-measure set of productivity histories.

in the contemporaneous cash flows  $\pi_t$ . It is then easy to see that any payment scheme with such properties implementing the effort and consumption policies  $(\xi, c)$  must satisfy, for any  $\theta$ ,

$$\begin{aligned}\frac{\partial x_1(\theta_1, \pi_1)}{\partial \pi_1} \Big|_{\pi_1 = \pi_1(\theta_1)} &= \psi'(\xi_1(\theta_1)) w'(v(c_1(\theta_1))) \\ \frac{\partial x_2(\theta, \pi_2)}{\partial \pi_2} \Big|_{\pi_2 = \pi_2(\theta)} &= \psi'(\xi_2(\theta)) w'(v(c_2(\theta)))\end{aligned}$$

with  $\pi_t(\theta) = \theta_t + \xi_t(\theta)$ ,  $t = 1, 2$ . Under such schemes, our definition of the power of incentives then coincides with the rate at which the period- $t$  compensation changes with the period- $t$  cash flows around the target level. This definition parallels the one typically given in the literature that restricts attention to linear schemes (see, e.g., Lazear (2000)), but with two important qualifications. First, our notion is a local measure. This is useful because, in general, compensation which is linear in cash flows may fail to implement the optimal policies  $(\xi^*, c^*)$ . Second, our definition does not require confining attention to differentiable schemes. This is convenient because we cannot rule out the possibility that, for certain specifications, the optimal policies  $(\xi^*, c^*)$  may be implementable only with non-differentiable schemes.<sup>25</sup>

With this discussion in mind, hereafter we interpret the left-hand sides of (7) and (8) as the ex-ante expected power of incentives under optimal schemes. Below we examine how these expectations depend on the persistence of the manager's productivity (here captured by  $\gamma$ ) and the manager's degree of risk aversion.

**Risk neutrality.** When the manager is risk neutral (when  $v$  is equal to the identity function), the power of incentives is simply the marginal disutility of effort, evaluated at the prescribed effort level  $\xi_t^*(\theta)$ . In this case, the Euler conditions (7) and (8) pin down not only the dynamics of the power of incentives, but also the dynamics of effort. In particular, when the disutility of effort is quadratic over the relevant range, as we have assumed, then

$$\mathbb{E} \left[ \xi_1^*(\tilde{\theta}_1) \right] = 1 - \mathbb{E} \left[ \frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \right] \text{ and} \quad (10)$$

$$\mathbb{E} \left[ \xi_2^*(\tilde{\theta}) \right] = 1 - \gamma \mathbb{E} \left[ \frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \right]. \quad (11)$$

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<sup>25</sup>However, we conjecture that differentiable schemes can always implement policies which are virtually optimal. By this we mean the following. Let  $(\xi^*, c^*)$  be fully optimal policies. For any  $\varepsilon > 0$  there exist policies  $(\xi, c)$  and a differentiable compensation scheme  $x$  such that the following are true: (i) the contract  $\Omega \equiv \langle \xi, x \rangle$  is individually rational and incentive compatible for the manager; (ii) in each state  $\theta$ , the compensation the manager receives under  $\Omega$  is given by  $c$ ; and (iii) with probability one  $\|(\xi(\theta), c(\theta)) - (\xi^*(\theta), c^*(\theta))\| \leq \varepsilon$ . In other words, the firm can always implement policies arbitrarily close to the fully-optimal ones using differentiable schemes. Moreover, we conjecture that, when the manager is risk averse, if the policies  $(\xi, c)$  yield profits arbitrarily close to the ones under the fully optimal policies, then  $(\xi, c)$  must be arbitrarily close to  $(\xi^*, c^*)$  in the  $\mathcal{L}_1$  norm. Virtually optimal policies can then be expected to inherit the same dynamic properties discussed below as the fully optimal policies. This is because the properties discussed below refer to the *expectation* of  $\psi'(\xi_t(\theta)) w'(v(c_t(\theta)))$ , where the expectation is over all possible productivity histories.

Conditions (10) and (11) highlight how the dynamics of expected effort depends on the persistence parameter  $\gamma$ : expected effort is higher in the second period when  $\gamma < 1$  and is the same in both periods when  $\gamma = 1$ . These conditions, and hence these predictions about the dynamics of effort, hold irrespective of the shape of the period-1 distribution  $F_1$ . In particular, they hold without the assumption that  $F_1$  is log-concave, a common restriction in the mechanism design literature.<sup>26</sup>

To interpret the result, note that the left-hand sides of Conditions (10) and (11) represent the expected marginal cost of higher effort, in terms of extra disutility for the manager. The right-hand side represents the expected marginal benefit for the firm (stemming from the increase in cash flows), less a term which captures the effect of higher effort on the surplus that the firm must leave to the manager to induce him to reveal his productivity (this surplus is over and above the minimal compensation required to compensate the manager for his disutility of effort, as one can see by inspecting (4)).

Now consider why, under risk neutrality, the expected power of incentives (and hence effort) increases over time. Recall, in particular, that the reason why the firm distorts downward the effort asked of those managers whose initial productivity is low is to reduce the rents it must leave to those managers whose initial productivity is high. When productivity is not fully persistent, these distortions are more effective in reducing managerial rents early in the relationship as opposed to later on. Distortions are therefore smaller at later dates, explaining why the expected power of incentives increases with tenure. The increase is most pronounced when productivity is least persistent. Indeed, as we approach the case where productivity is independent over time (i.e., when  $\gamma$  is close to zero), the expected effort the firm asks of each manager in the second period is close to the first-best level ( $e^{FB} = 1$ ).

**Risk aversion.** To understand how risk aversion affects the above conclusions, it is useful to start with utility functions of the isoelastic form  $v_\rho(c) = \frac{(c+z)^{1-\rho}}{1-\rho}$  for  $z, \rho \geq 0$ . We view  $z$  as a level of consumption guaranteed by other income sources, so that the manager has constant-relative-risk-aversion preferences. We consider  $\rho \leq \bar{\rho}$  for some  $\bar{\rho} < +\infty$ , and assume that  $z$  is large enough to guarantee that  $c + z$  remains positive over these values of  $\rho$ , under the optimal policies.

These felicity functions offer a convenient parameterization of the degree of risk aversion which we use in the next result to show how the above observations for a risk-neutral manager extend to sufficiently low degrees of risk aversion.<sup>27</sup> Our key finding, however, is Proposition 3 below, which applies to arbitrary utility functions.

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<sup>26</sup>The reason why here we could dispense with such an assumption is that our result applies to the dynamics of the average effort as opposed to the dynamics of effort along any productivity history.

<sup>27</sup>This result is similar in spirit to Proposition 3 in Battaglini and Coate (2008), which (in a public finance setting) identifies a form of continuity of the optimal contract in the risk-aversion parameter in a neighborhood of risk neutrality.

**Proposition 2** *Fix the productivity distributions  $F_1$  and  $G$  and the disutility of effort function  $\psi$ . Suppose that the manager’s preferences for consumption in each period are represented by the isoelastic felicity function  $v_\rho(\cdot)$  defined above. For any level of persistence  $\gamma < 1$  of the manager’s productivity, there exists a critical level of risk aversion  $\bar{\rho} > 0$  such that, for all  $\rho \in [0, \bar{\rho}]$ , the expected power of incentives under any optimal compensation scheme is higher in period two than in period one.*

The levels of risk aversion for which the result in Proposition 2 holds (i.e., how large one can take  $\bar{\rho}$ ) naturally depends on the persistence of initial productivity  $\gamma$ . For a fixed level of risk aversion, if  $\gamma$  is close to 1, i.e., if the initial productivity is highly persistent, then the above result about the dynamics of the power of incentives is *completely reversed*: the power of incentives declines, on average, over time, as stated in the next result.

**Proposition 3** *Fix the productivity distributions  $F_1$  and  $G$  and the disutility of effort function  $\psi$ . For any strictly concave felicity function  $v$ , there exists a critical level of persistence  $\bar{\gamma} \leq 1$  such that, for all  $\gamma \geq \bar{\gamma}$ , the expected power of incentives under any optimal compensation scheme is weakly higher in period one than in period two. Provided that  $c_2^*(\theta)$  varies with  $\theta_2$  over a positive-probability set of  $\theta_1$ , then  $\bar{\gamma}$  is strictly less than 1 and the expected power of incentives is strictly lower in period two than in period one.<sup>28</sup>*

To understand the result, recall that, when the manager is risk neutral and  $\gamma$  is close to 1, then the effect of higher period-2 effort on the surplus that the firm must leave to each manager is almost the same as the effect of higher period-1 effort (it is exactly the same when  $\gamma = 1$ ). As a result, the power of incentives, on average, is almost the same across the two periods. If the manager is risk averse, however, then incentivizing a high level of effort at date 2 is more costly for the firm. Incentivizing high effort must expose the manager to volatile compensation as a result of his own private uncertainty about date-2 productivity. Since the manager dislikes this volatility, he must be given additional compensation by the firm. This suggests the firm should lower the effort asked at date 2 to save on managerial compensation.

To see this more formally, consider the Euler conditions (7) and (8) in Proposition 1 and focus on the case where productivity is fully persistent (i.e.,  $\gamma = 1$ ). When the disutility of effort is quadratic, the first two terms in the right-hand sides of these equations are identical. The key difference across the two periods comes from the third term in the right-hand side of (8) which is always negative and captures the effect of the volatility in the period-2 compensation on the surplus that the firm must give to the manager to induce him to participate. This volatility originates in the need to make period-2 compensation sensitive to period-2 performance to incentivize period-2 effort. As a result,

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<sup>28</sup>We expect that this condition holds in all but “knife-edge” cases. A sufficient condition, for instance, is that the hazard rate  $\frac{f_1(\theta_1)}{1-F_1(\theta_1)}$  is increasing and that the manager’s degree of risk aversion is not too large.

the volatility can be reduced by lowering the power of incentives in period two. Under any optimal contract, the firm thus reduces the power of incentives over time to reduce the manager’s exposure to compensation risk. The same conclusions apply to the case where productivity is highly but not fully persistent. Once again, how persistent productivity must be for the result to obtain depends on the details of the productivity distribution (that is on  $G$  and  $F_1$ ) and on the degree of managerial risk aversion (that is, on  $v$ ).

One further way to understand why the expected power of incentives declines over time when the manager is risk averse and productivity is sufficiently persistent is as follows. Suppose that period-2 effort is restricted to depend only on period-1 productivity (that is, suppose both  $\xi_1$  and  $\xi_2$  depend only on  $\theta_1$ ). The manager’s period-2 compensation can then be written as

$$w \left( \begin{array}{l} \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta_1)) + \int_{\theta_1}^{\theta_1} \{\psi'(\xi_1(s)) + \gamma\psi'(\xi_2(s))\} ds \\ + \left(\theta_2 - \mathbb{E}^{\tilde{\theta}_2|\theta_1}[\tilde{\theta}_2]\right) \psi'(\xi_2(\theta_1)) - v(c_1(\theta_1)) \end{array} \right).$$

It is then easy to see that the volatility of the period-2 compensation is increasing in the period-2 effort  $\xi_2(\theta_1)$ . When the manager is risk averse,  $w$  is strictly convex. By reducing  $\xi_2$ , the firm then reduces the expected period-2 compensation, for any level of the period-1 productivity.<sup>29</sup> When this effect is strong, the firm may then find it optimal to reduce the power of incentives over time.

#### 4.4 Further discussion of optimal policies

Conditions (7) and (8) yield important insights about the effect of seniority on the power of incentives under optimal contracts. These conditions were obtained by maximizing the firm’s profits over all *implementable* policies. As noted above, an alternative (and more canonical) approach involves maximizing the firm’s profits subject only to certain “local incentive constraints”. In our environment, this amounts to maximizing (5) over all possible effort and compensation policies, thus ignoring the possibility that policies that maximize (5) need not be implementable by a contract which is individually rational and incentive compatible for the manager. This second approach is called the “relaxed approach”. Whether this relaxed approach yields policies that can indeed be implemented under an incentive-compatible contract is something that is verified ex-post, once the solution to maximizing (5) is in hand. One advantage of this approach is that (when validated) it facilitates a more precise characterization of the optimal policies. In our environment, this simply

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<sup>29</sup>If we restrict attention to effort policies that depend only on period-1 productivity, then the result in Proposition 3 applies not only to the dynamics of the power of incentives but also to the dynamics of expected effort: i.e., expected effort declines over time under the assumptions of the proposition. When we do not impose this restriction, however, we have been unable to disentangle the effect of risk aversion on expected effort from its effect on the expected power of incentives. This appears difficult because of the need to control for the correlation between second-period compensation and second-period effort, conditional on the period-1 productivity.

means that one can derive conditions analogous to (7) and (8) which hold ex-post, i.e. for each possible productivity history, as opposed to in expectation.

**Proposition 4** *Let  $\xi^R$  and  $c^R$  be effort and compensation policies that maximize (5) (note that these policies need not be sustainable under an incentive-compatible contract). Then, with probability one,  $\xi^R$  and  $c^R$  must satisfy Conditions (9) and (2). In addition,  $\xi^R$  and  $c^R$  must satisfy the following two conditions:<sup>30</sup>*

$$\psi'(\xi_1^R(\theta_1)) w'(v(c_1^R(\theta_1))) = 1 - \frac{\psi''(\xi_1^R(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'(v(c_1^R(r))) f_1(r) dr, \quad (12)$$

and

$$\begin{aligned} \psi'(\xi_2^R(\theta)) w'(v(c_2^R(\theta))) &= 1 - \gamma \frac{\psi''(\xi_2^R(\theta))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'_1(v(c_1^R(r))) f_1(r) dr \\ &\quad - \frac{\psi''(\xi_2^R(\theta))}{f_2(\theta_2|\theta_1)} \int_{\theta_2}^{\bar{\theta}_2} \{w'(v(c_2^R(\theta_1, r))) - w'(v(c_1^R(\theta_1)))\} f_2(r|\theta_1) dr. \end{aligned} \quad (13)$$

The policy  $\xi^R$  is essentially unique. If  $v$  is strictly concave, then  $c^R$  is also essentially unique.

Note that, when the manager is risk neutral and the disutility of effort is quadratic, the policy  $\xi^R$  is given by

$$\xi_1^R(\theta) = 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \quad (14)$$

$$\xi_2^R(\theta) = 1 - \gamma \frac{1 - F_1(\theta_1)}{f_1(\theta_1)}. \quad (15)$$

If the hazard rate  $\frac{f_1(\theta_1)}{1 - F_1(\theta_1)}$  is increasing, as is often assumed in the literature (and as satisfied by a wide class of distributions), then one can establish the existence of compensation schemes that implement the above policies. Indeed, it is enough to consider schemes which pay the manager as a linear function of the cash flows (see the working paper version for details). Furthermore, in this case, managers whose initial productivity is high are offered higher powered incentives than managers whose initial productivity is low. The reason for this result relates once again to the effect of effort on managerial rents. When the hazard rate of the period-1 distribution is increasing, the weight that the firm assigns to rent extraction relative to efficiency (as captured by the inverse hazard rate  $[1 - F_1(\theta_1)]/f_1(\theta_1)$ ) is smaller for higher types (recall that asking type  $\theta_1$  to exert more effort requires increasing the rent of all types  $\theta'_1 > \theta_1$ ). As a result, the firm offers higher powered incentives to those managers whose initial productivity is high. When it comes to the dynamics of the power of incentives, we then have the following comparison across types.

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<sup>30</sup>Again, these conditions must hold with probability one.

**Corollary 1** *Suppose that the manager is risk neutral and that the hazard rate of the period-1 distribution is increasing. Then the increase in the power of incentives over time is larger for those managers whose initial productivity is low.*

The result reflects the fact that period-1 effort is more distorted for those managers whose initial productivity is low, implying that, over time, the correction is larger for those types. The result in the previous corollary thus yields another testable prediction: because productivity is positively correlated with performance, the econometrician should expect to find a negative relationship between early performance and the increase in the power of incentives over time. Note that this prediction is not shared by any of the alternative theories mentioned in the Introduction which explain increases in the power of incentives over time.

Next, consider the case of a risk-averse manager. In this case, verifying that the policies  $(\xi^R, c^R)$  that solve the relaxed program can be implemented under a contract that is incentive compatible for the manager (and hence that such policies are sustained under an optimal contract) is more difficult. We do so for numerical examples on a case-by-case basis. To illustrate, consider a manager with CRRA preferences with risk aversion parameter  $\rho = 1/2$  and assume that  $z = 0$  (meaning that  $v_\rho(c) = 2\sqrt{c}$ ). Further assume that  $\theta_1$  is uniformly distributed over  $[0, 1/2]$  and  $\varepsilon$  is uniformly distributed over  $[0, 1]$ . Figures 1 and 2 show, for different levels of productivity persistence ( $\gamma = 1$  and  $\gamma = 1/2$ ), how the power of incentives in period 1 and the expected power of incentives in period 2 vary with the initial productivity  $\theta_1$ .

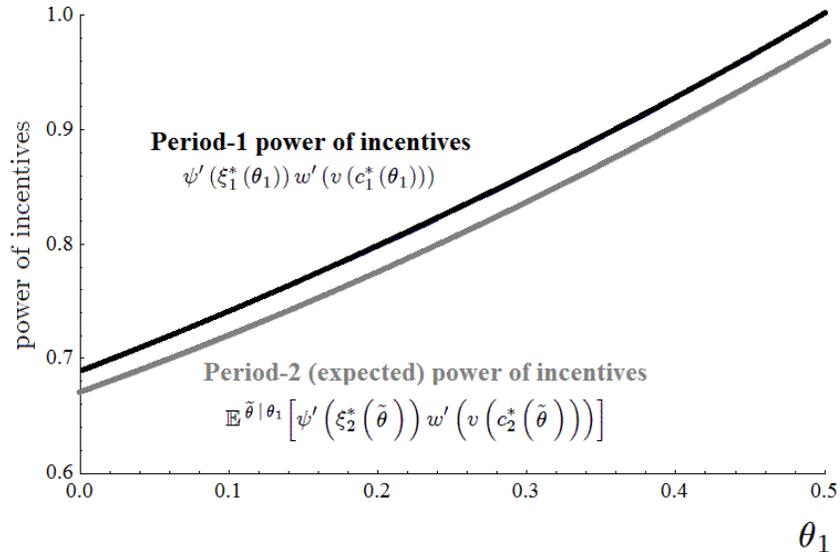


Figure 1: Dynamics of power of incentives:  $\gamma = 1$

When productivity is fully persistent (i.e., for  $\gamma = 1$  as in Figure 1), the power of incentives in period 1 is higher than the expected power of incentives in period 2, irrespective of the initial

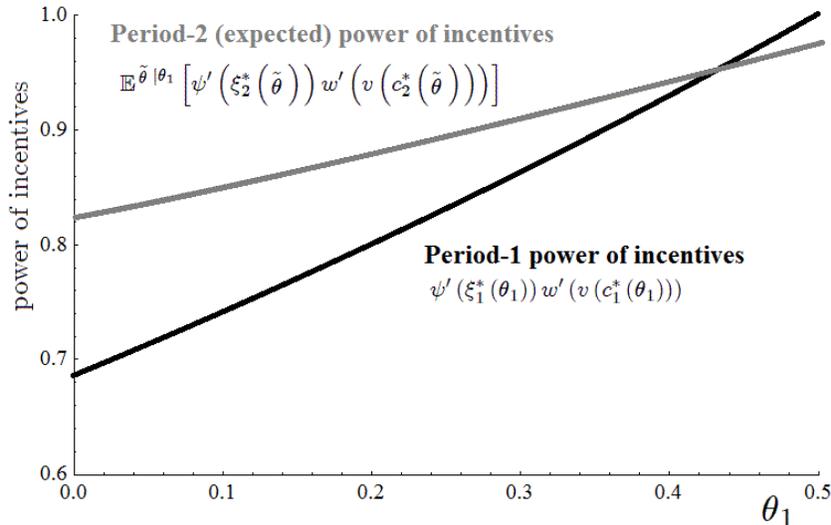


Figure 2: Dynamics of power of incentives:  $\gamma = 1/2$

productivity. This result thus parallels the one in Proposition 3 but without averaging across different productivity levels. For smaller values of  $\gamma$  (e.g., for  $\gamma = 1/2$  as in Figure 2), whether the power of incentives is expected to increase or decrease over time depends on the initial productivity. For high initial productivities, the power of incentives declines over time, whereas the opposite is true for lower productivity levels.

These findings reflect the trade-off between reducing the manager's exposure to risk, which calls for reducing both the power of incentives and effort at later periods, and reducing the manager's expected rents, which calls for low-powered incentives early on followed by higher-powered incentives later in the relationship. The effect of the power of incentives on expected rents is similar across the two periods when either (i) productivity is persistent ( $\gamma = 1$ ), or (ii) the initial productivity is high, in which case the effect of effort on rents is negligible. In these cases, the firm optimally reduces the power of incentives over time so as to reduce the risk the manager faces when it comes to his future compensation.

Next consider the period-1 effort level and the expected period-2 effort level conditional on the period-1 productivity, as in Figures 3 and 4. Whether effort increases or decreases over time, conditional on the initial productivity, follows a similar pattern as for the power of incentives. However, there can be qualitative differences between the power of incentives and the effort policies. For instance, when  $\gamma = 1/2$  (as in Figure 4), the expected period-2 effort is decreasing in the initial productivity, whereas the expected power of incentives is increasing. The reason for this possibility is that, when the manager has a higher period-1 productivity, he expects to be paid more, and is therefore more difficult to incentivize. That is, for the same power of incentives, the effort the manager is willing to choose is lower because his marginal value for additional consumption is lower.

This illustrates the more general principle that, for a risk-averse manager, the power of incentives and effort, while related, must be considered separately.

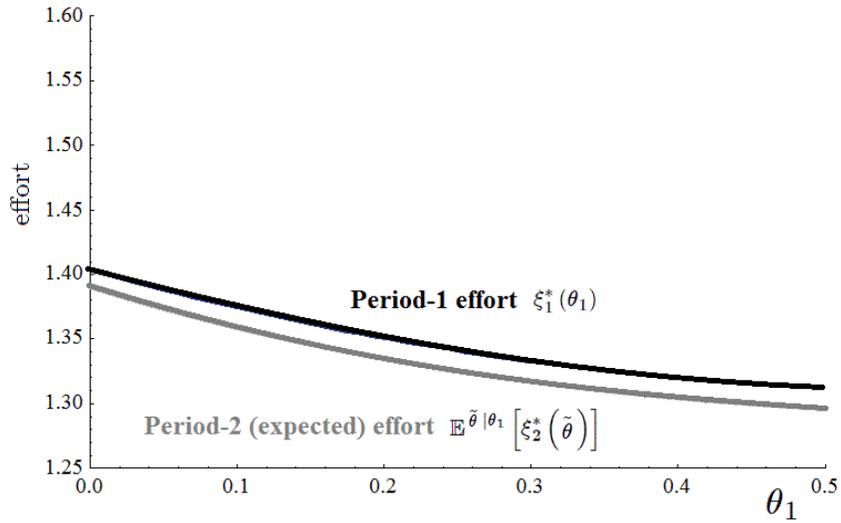


Figure 3: Dynamics of effort:  $\gamma = 1$

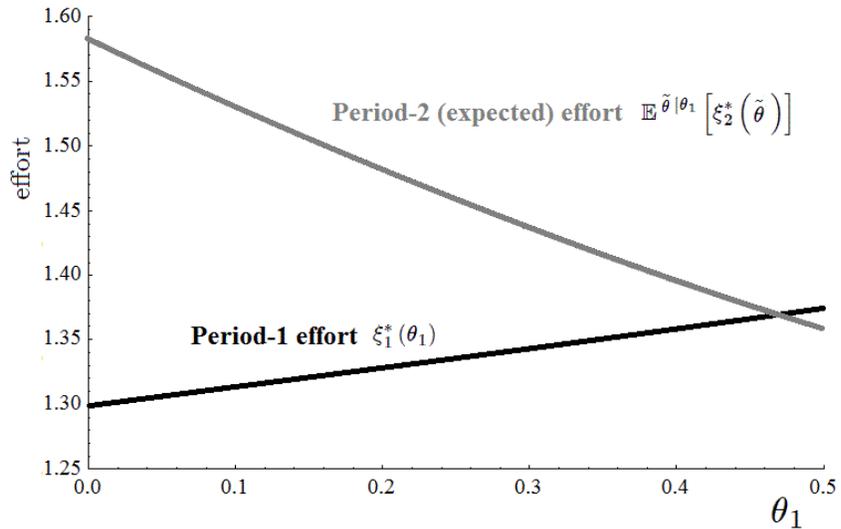


Figure 4: Dynamics of effort:  $\gamma = 1/2$

## 5 Conclusions

We investigated the optimal dynamics of incentives for a manager whose ability to generate profits for the firm changes stochastically over time. In doing so, we appealed to a definition of the “power of incentives” which seems appropriate for settings where payment schemes need not be linear (or even differentiable) in the cash flows.

When the manager is risk neutral, we showed that it is typically optimal for the firm to offer a compensation scheme where the power of incentives increases, on average, over time, thus inducing the manager to exert more effort as his tenure in the firm grows. We then showed how risk aversion can reverse the profitability of such schemes.

In future work, it would be interesting to calibrate the model so as to quantify the relevance of the effects identified in the paper and to derive specific predictions about the combination of stocks, options, and fixed pay that implement the optimal dynamics of incentives.

We conclude with a few remarks about the applicability of the approach we followed in the present paper (which involves tackling the full program directly) to richer specifications of the contracting problem. Euler equations similar to those in Proposition 1 can be obtained in settings with arbitrarily many periods and richer stochastic processes. When the manager is risk neutral, these equations immediately yield a closed-form expression for expected effort in each period (analogous to Equations 10 and 11 in the paper). Interestingly, these dynamics can be obtained without any of the conditions typically imposed in the dynamic contracting literature (e.g., log-concavity of the period-1 distribution, monotonicity of the impulse responses of future types to the initial ones). This is because the predictions identified by this approach apply to the “average” dynamics, where the average is over all possible realizations of the type process, as opposed to ex-post. When the manager is risk-averse, these Euler equations are more convoluted but suggest that insights similar to those in Propositions 2 and 3 should continue to hold.

On the other hand, the assumption that the disutility of effort is quadratic is more difficult to relax. This assumption plays no role in the traditional approach (consisting in solving a “relaxed program” and then validating its solution). However, when tackling directly the “full program” this assumption permitted us to identify a simple class of perturbations that preserve incentive compatibility which we then used to arrive at the Euler equations in Proposition 1. In this respect, this assumption plays in our environment a role similar to that of the linearity of payoffs in Rochet and Choné’s (1998) analysis of multidimensional screening. While the approach can be adapted to accommodate for more general specifications, the Euler-type conditions that emerge under richer specifications appear less amenable to tractable analysis.

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## Appendix

**Proof of Lemmas 1 and 2.** The proof for both results follows from the following observation. A necessary condition for a contract  $\Omega = \langle \xi, x \rangle$  to be incentive compatible for the manager is that the manager prefers to follow a truthful and obedient strategy in each period, rather than lying about his productivity and then adjusting his effort choice so as to hide the lie (i.e., so as to generate the same cash flows as the type being mimicked). This step turns the problem into one of pure adverse selection, as first suggested by Laffont and Tirole (1986) in a static setting. Using results from the recent dynamic mechanism design literature, one can then show that, when the payoff structure satisfies the properties in the model setup (in particular, when the disutility of effort is differentiable and Lipschitz continuous), then a necessary condition for incentive compatibility is that, for any  $(\theta_1, \theta_2)$ , the manager's ex-post payoff satisfies

$$V(\theta_1, \theta_2) = V(\theta_1, \underline{\theta}_2) + \int_{\underline{\theta}_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds \quad (16)$$

and that his period-1 expected payoff satisfies (4). The condition (16) is analogous to the static one in Laffont and Tirole (1986). The necessity of (4) follows from adapting to the environment under examination the result in Theorem 1 in Pavan, Segal, and Toikka (2012)—see also our previous work, Garrett and Pavan (2012) for a derivation of a similar condition in a model of managerial turnover.

Combining (16) with (4) then implies that, under any contract that is individually rational and incentive compatible, with probability one (i.e., except possibly over a zero-measure set of productivity histories), the utility that each manager derives from his lifetime compensation must satisfy (2) with  $K = \mathbb{E}^{\tilde{\theta}|\underline{\theta}_1}[V(\tilde{\theta})] \geq 0$ , as stated in Lemma 1. That the principal's payoff in turn satisfies the representation in Lemma 2 follows from the observations above along with the fact that, under any optimal contract,  $K = 0$ . Q.E.D.

**Proof of Proposition 1.** The proof is in three steps. Step 1 identifies necessary and sufficient conditions for any given effort policy to be implementable. Step 2 identifies a family of perturbations that preserve incentive compatibility and then uses this family to identify necessary conditions for the proposed effort and compensation policies  $(\xi^*, c^*)$  to be sustained under an optimal contract. Finally, Step 3 establishes our result on the uniqueness of  $(\xi^*, c^*)$ .

### Step 1 (Necessary and sufficient conditions for implementability of any effort policy).

We start with the following lemma:

**Lemma 3** *The following conditions are necessary and sufficient for an effort policy  $\xi$  to be implementable: (a) for all  $\theta_1, \hat{\theta}_1 \in \Theta_1$ ,*

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left( \xi_1(\hat{\theta}_1) - (s - \hat{\theta}_1) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi' \left( \xi_2(\hat{\theta}_1, \tilde{\theta}_2) \right) \right] \right\} ds \\ & \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left( \xi_1(s) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi' \left( \xi_2(s, \tilde{\theta}_2) \right) \right] \right\} ds, \end{aligned} \quad (17)$$

and (b) for all  $\theta_1$ ,  $\pi_2(\theta_1, \theta_2) \equiv \xi_2(\theta_1, \theta_2) + \theta_2$  is nondecreasing in  $\theta_2$ .

**Proof.** Given the policies  $(\xi, c)$ , let  $x$  be the compensation scheme defined as follows. In each period  $t$ , given the reports  $\theta$ , the manager is assigned a “target”  $\pi_t(\theta) = \theta_t + \xi_t(\theta)$ . He is then paid a fixed compensation equal to  $c_t(\theta)$  if  $\pi_t \geq \pi_t(\theta)$  and otherwise is charged a large fine. It is easy to see that any pair of policies  $(\xi, c)$  that is implementable under some compensation scheme  $\hat{x}$  is also implementable under the compensation scheme  $x$  defined above. Hereafter, we thus confine attention to contracts where the compensation scheme satisfies this property.

Next note that the following are necessary and sufficient conditions for the manager to find it optimal to follow a truthful and obedient strategy in period two, irrespective of the period-1 type and the period-1 cash flows (see, in particular, Laffont and Tirole, 1986): (a) for any  $\theta_1$ ,  $\pi_2(\theta_1, \theta_2) \equiv \xi_2(\theta_1, \theta_2) + \theta_2$  is nondecreasing in  $\theta_2$ , and (b) for any  $\theta = (\theta_1, \theta_2)$ ,

$$c_2(\theta) = w \left( \psi(\xi_2(\theta)) + K_2(\theta_1) + \int_{\theta_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds \right) \quad (18)$$

where  $K_2(\cdot)$  is an arbitrary function of  $\theta_1$ .

Now consider any contract  $\Omega$  such that (i) the manager’s equilibrium payoff satisfies the conditions in Lemma 1 (recall that these conditions are necessary for incentive compatibility; also observe that these conditions imply property (18) above), and (ii)  $\pi_2(\theta)$  is nondecreasing in  $\theta_2$ , for any  $\theta_1$ . Given this contract, let  $V^\Omega(\theta_1, \hat{\theta}_1)$  be the payoff that a manager whose period-1 productivity is  $\theta_1$  obtains when he reports  $\hat{\theta}_1$ , then chooses period-1 effort optimally (which means attaining the target  $\pi_1(\hat{\theta}_1)$ ), and then behaves optimally in period 2 (which means following a truthful and obedient strategy). Then

$$\begin{aligned} V^\Omega(\theta_1, \hat{\theta}_1) &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \psi(\xi_1(\hat{\theta}_1)) - \psi \left( \xi_1(\hat{\theta}_1) - (\theta_1 - \hat{\theta}_1) \right) \\ &\quad + \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ \int_{\theta_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] - \mathbb{E}^{\tilde{\theta}_2|\hat{\theta}_1} \left[ \int_{\theta_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] \\ &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \psi' \left( \xi_1(\hat{\theta}_1) - (s - \hat{\theta}_1) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi'(\xi_2(\hat{\theta}_1, \tilde{\theta}_2)) \right] ds. \end{aligned}$$

Because the contract  $\Omega$  satisfies the conditions in Lemma 1

$$V^\Omega(\theta_1, \theta_1) = V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \left[ \psi'(\xi_1(s)) + \mathbb{E}^{\tilde{\theta}_2|s} \left[ \psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right] ds.$$

It follows that a necessary and sufficient condition for  $V^\Omega(\theta_1, \hat{\theta}_1) \leq V^\Omega(\theta_1, \theta_1)$  for all  $\theta_1, \hat{\theta}_1$  is that  $\xi$  satisfies the integral monotonicity condition (17). Combining the above results, we then have that the conditions in the lemma are both necessary and sufficient for the policy  $\xi$  to be implementable. ■

**Step 2 (Euler Equations).** We now establish that the Euler Conditions (7), (8), and (9) are necessary conditions for the policies  $\xi^*$  and  $c^*$  to be implemented under an optimal contract. To see this, consider the perturbed effort policy  $\xi = (\xi_1^*(\cdot) + a, \xi_2^*(\cdot) + b)$  for some constants  $a, b \in \mathbb{R}$ . Then consider the perturbed compensation policy  $c$  given by  $c_1(\theta_1) = c_1^*(\theta_1)$  and  $c_2(\theta) = w(W(\theta; \xi) - v(c_1^*(\theta_1)))$  all  $\theta$ . It is easy to see that, if the policy  $\xi^*$  satisfies the conditions of Lemma 3, so does the perturbed policy  $\xi$ . Furthermore, because the compensation policy  $c$  is constructed so that the manager's payoff under a truthful and obedient strategy continues to satisfy the conditions in Lemma 1, it is easy to see that, if the policies  $\xi^*$  and  $c^*$  are implementable, so are the perturbed policies  $(\xi, c)$ .

Now consider the firm's expected profits under the perturbed policies. For the original policies to be optimal, the expected profits must be maximized at  $a = b = 0$ . Using (5), we then have that the derivative of the firm's profits with respect to  $a$ , evaluated at  $a = b = 0$ , vanishes only if the policies  $\xi^*$  and  $c^*$  satisfy Condition (7) (to see this, it suffices to take the derivative of  $\mathbb{E}[U^P]$  with respect to  $a$  and then integrate by parts). Likewise, the derivative of  $\mathbb{E}[U^P]$  with respect to  $b$ , evaluated at  $a = b = 0$ , vanishes only if the policies satisfy (8).

The argument for the necessity of (9) is similar. Fix the effort policy  $\xi^*$  and consider a perturbation of the period-1 consumption policy so that the new policy satisfies  $v(c_1(\theta_1)) = v(c_1^*(\theta_1)) + a\eta(\theta_1)$  for a scalar  $a$  and some measurable function  $\eta(\cdot)$ . In other words,  $c_1(\theta_1) = w(v(c_1^*(\theta_1)) + a\eta(\theta_1))$ . Then adjust the period-2 compensation so that  $c_2(\theta) = w(W(\theta; \xi^*) - v(c_1(\theta_1)))$  all  $\theta$ . It is easy to see that the pair of policies  $(\xi^*, c)$  continues to be implementable. The firm's expected profits under the perturbed policies are

$$\mathbb{E}[U^P] = \mathbb{E} \left[ \begin{array}{c} \tilde{\theta}_1 + \xi_1^*(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2^*(\tilde{\theta}) \\ -w(v(c_1^*(\tilde{\theta}_1)) + a\eta(\tilde{\theta}_1)) - w(W(\tilde{\theta}; \xi^*) - v(c_1^*(\tilde{\theta}_1)) - a\eta(\tilde{\theta}_1)) \end{array} \right].$$

Optimality of  $c^*$  then requires that the derivative of this expression with respect to  $a$  vanishes at  $a = 0$  for all measurable functions  $\eta$ . This is the case only if Condition (9) holds.

**Step 3 (Uniqueness of the optimal policies).** We first show that the optimal effort policy is essentially unique (i.e., unique up to a zero-measure set of productivity histories). Suppose, towards a contradiction, that there exist two optimal contracts  $\Omega^\#$  and  $\Omega^{\#\#}$  implementing the policies  $(\xi^\#, c^\#)$  and  $(\xi^{\#\#}, c^{\#\#})$  respectively. Now suppose that  $\xi^\#$  and  $\xi^{\#\#}$  prescribe different effort levels over a set of productivity histories of strictly positive (probability) measure. Let  $\alpha \in (0, 1)$  and define

$\xi^\alpha \equiv \alpha \xi^\# + (1 - \alpha) \xi^{\#\#}$ ,  $c_1^\alpha(\theta_1) = \alpha c_1^\#(\theta_1) + (1 - \alpha) c_1^{\#\#}(\theta_1)$  and  $c_2^\alpha(\theta) \equiv w(W(\theta; \xi^\alpha) - v(c_1^\alpha(\theta_1)))$ . Note that the pair of policies  $(\xi^\alpha, c^\alpha)$  is implementable (to see this, note that (i) the effort policy  $\xi^\alpha$  satisfies the conditions of Lemma 3, and (ii) the equilibrium payoffs under the policies  $(\xi^\alpha, c^\alpha)$  satisfy the condition in Lemma 1).

Next, note that (5) is strictly concave in the effort policy  $\xi$  (recognizing that the policy  $\xi$  enters (5) also through  $W(\theta; \xi)$ , as defined in (3)) and weakly concave in  $c_1$ .<sup>31</sup> This means that the firm's expected profits  $\mathbb{E}[U^P]$  under the new policy  $(\xi^\alpha, c^\alpha)$  are strictly higher than under either  $(\xi^\#, c^\#)$  or  $(\xi^{\#\#}, c^{\#\#})$ , contradicting the optimality of these policies.

By the same arguments, if  $v$  is strictly concave and  $c_1^\#(\theta_1) \neq c_1^{\#\#}(\theta_1)$  over a set of positive probability measure, then the new policies  $(\xi^\alpha, c^\alpha)$  constructed above yield strictly higher profits than those sustained under  $\Omega^\#$  and  $\Omega^{\#\#}$ , irrespective of whether or not  $\xi^\# \neq \xi^{\#\#}$ . This in turn implies that, when  $v$  is strictly concave, the optimal compensation policy is also (essentially) unique. Q.E.D.

**Proof of Proposition 2.** Let  $\xi_\rho^* \equiv (\xi_{\rho,1}^*, \xi_{\rho,2}^*)$  be the (essentially unique) effort policy sustained under any optimal contract, when the manager's degree of relative risk aversion is  $\rho$ . Likewise, let  $c_\rho^* \equiv (c_{\rho,1}^*, c_{\rho,2}^*)$  be a consumption policy sustained under an optimal contract and recall that such a policy is essentially unique when  $\rho > 0$ , i.e., when the manager is strictly risk averse.

Suppose, towards a contradiction, that the result in the proposition is not true. This means that, for any  $n \in \mathbb{N}$ , there exists  $\rho_n \in (0, \frac{1}{n})$  such that

$$\mathbb{E} \left[ \psi' \left( \xi_{\rho_n,1}^*(\tilde{\theta}_1) \right) w'_{\rho_n} \left( v_{\rho_n}(c_{\rho_n,1}^*(\tilde{\theta}_1)) \right) \right] \geq \mathbb{E} \left[ \psi' \left( \xi_{\rho_n,2}^*(\tilde{\theta}) \right) w'_{\rho_n} \left( v_{\rho_n}(c_{\rho_n,2}^*(\tilde{\theta})) \right) \right],$$

where  $w_{\rho_n}$  is the inverse of  $v_{\rho_n}$ . On the other hand, we have that, when  $\rho = 0$  (that is, when the manager is risk neutral),  $\mathbb{E} \left[ \psi' \left( \xi_{0,1}^*(\tilde{\theta}_1) \right) \right] < \mathbb{E} \left[ \psi' \left( \xi_{0,2}^*(\tilde{\theta}) \right) \right]$ , as one can easily see from (10) and (11). Using (9), one can also easily see that the compensation policies  $c_{\rho,1}^*$  and  $c_{\rho,2}^*$  are bounded uniformly (almost surely, i.e. except possibly over a zero-probability set of productivity histories) over positive  $\rho$  in a neighborhood of zero. By our assumption that optimal effort is bounded from above by  $\bar{e}$ , the following must then be true: there exist  $\varepsilon > 0$  and  $N \in \mathbb{N}$  such that, for all  $n \geq N$ ,  $\Pr \left\{ \left\| \xi_{\rho_n}^*(\tilde{\theta}) - \xi_0^*(\tilde{\theta}) \right\| \geq \varepsilon \right\} \geq \varepsilon$ .

Denote by  $h(\xi)$  the principal's expected profits (5) when the manager is risk neutral (i.e., when  $\rho = 0$ ) and the effort policy is  $\xi$  (recall that the principal's profits are uniquely determined by the effort policy when the manager is risk neutral, as one can see from (6)). The convexity of  $\psi$  (i.e., of  $e^2/2$ ) over effort levels in the relevant range  $[0, \bar{e}]$  implies the existence of a function  $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$  such that the following holds. If  $\xi'$  and  $\xi''$  are two policies taking values in  $(0, \bar{e})$  and satisfying  $\Pr \left\{ \left\| \xi'(\tilde{\theta}) - \xi''(\tilde{\theta}) \right\| \geq \varepsilon \right\} \geq \varepsilon$  for some  $\varepsilon > 0$ , then  $h\left(\frac{1}{2}\xi' + \frac{1}{2}\xi''\right) -$

<sup>31</sup>By *strict* concavity we mean with respect to the equivalence classes of functions which are equivalent if they are equal almost surely.

$(\frac{1}{2}h(\xi') + \frac{1}{2}h(\xi'')) > \kappa(\varepsilon)$ . Hence, there exists  $\varepsilon > 0$  and  $N \in \mathbb{N}$  such that, for all  $n \geq N$ ,  $h(\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*) - (\frac{1}{2}h(\xi_{\rho_n}^*) + \frac{1}{2}h(\xi_0^*)) > \kappa(\varepsilon)$ .

Next, continuity of the principal's expected profits (5) in the risk-aversion parameter  $\rho$  allows us to conclude that  $h(\xi_{\rho_n}^*) \rightarrow h(\xi_0^*)$  as  $n \rightarrow +\infty$ . Therefore,

$$h(\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*) > \frac{1}{2}h(\xi_{\rho_n}^*) + \frac{1}{2}h(\xi_0^*) + \kappa(\varepsilon) > h(\xi_0^*),$$

whenever  $n$  is sufficiently large. Using Lemma 3, for any  $n$ ,  $\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*$  must be an implementable effort policy. That  $h(\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*) > h(\xi_0^*)$  therefore contradicts the optimality of  $\xi_0^*$  for a risk-neutral manager. Q.E.D.

**Proof of Proposition 3.** We first show that the third term on the right-hand side of (8), which is given by

$$\mathbb{E} \left[ \frac{1}{f_2(\bar{\theta}_2|\bar{\theta}_1)} \int_{\bar{\theta}_2}^{\bar{\theta}_2} \left\{ w'(v(c_2^*(\tilde{\theta}_1, r))) - w'(v(c_1^*(\tilde{\theta}_1))) \right\} f_2(r|\tilde{\theta}_1) dr \right], \quad (19)$$

is negative. We establish this by showing that, with probability one (that is, for all but a zero-measure set of  $\theta$ ),

$$m(\theta_2; \theta_1) \equiv \int_{\theta_2}^{\bar{\theta}_2} \left\{ w'(v(c_2^*(\theta_1, r))) - w'(v(c_1^*(\theta_1))) \right\} f_2(r|\theta_1) dr \geq 0. \quad (20)$$

To see this, note that

$$\frac{\partial m(\theta_2; \theta_1)}{\partial \theta_2} = - \left( w'(v(c_2^*(\theta_1, \theta_2))) - w'(v(c_1^*(\theta_1))) \right) f_2(\theta_2|\theta_1). \quad (21)$$

Next, recall from (9) that, with probability one,  $w'(v(c_1^*(\theta_1))) = \mathbb{E}^{\bar{\theta}_2|\theta_1} \left[ w'(v(c_2^*(\theta_1, \tilde{\theta}_2))) \right]$ . Moreover,  $c_2^*(\theta_1, \cdot)$  must be nondecreasing (this follows from the fact that incentive compatibility requires that  $\pi_2(\theta_1, \cdot)$  be nondecreasing, as established in Lemma 3). Therefore, there exists  $\hat{\theta}_2(\theta_1) \in [\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$  such that  $c_2^*(\theta_1, \theta_2) \leq c_1^*(\theta_1)$  for  $\theta_2 \leq \hat{\theta}_2(\theta_1)$  and  $c_2^*(\theta_1, \theta_2) > c_1^*(\theta_1)$  for  $\theta_2 > \hat{\theta}_2(\theta_1)$ . Using that  $w'(v(\cdot))$  is increasing, together with (21), the function  $m(\cdot; \theta_1)$  must therefore be quasi-concave on  $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ . Finally, note that  $w'(v(c_1^*(\theta_1))) = \mathbb{E}^{\bar{\theta}_2|\theta_1} \left[ w'(v(c_2^*(\theta_1, \tilde{\theta}_2))) \right]$  implies  $m(\underline{\theta}_2(\theta_1); \theta_1) = 0$ . We have thus obtained that  $m(\underline{\theta}_2(\theta_1); \theta_1) = m(\bar{\theta}_2(\theta_1); \theta_1) = 0$ , as well as that  $m(\cdot; \theta_1)$  is quasi-concave, which together establish the claim in (20). Similarly, it is easy to see that the inequality in (20) is strict, unless  $c_2^*(\theta_1, \cdot)$  is constant over  $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ .

We can therefore conclude that the third term on the right-hand side of (8) is negative (and strictly negative unless, with probability one,  $c_2^*(\theta_1, \cdot)$  is constant over  $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ ).

Now consider the case where  $\gamma = 1$ . Since  $\psi'' = 1$ , the right-hand sides of (7) and (8) differ only by the third term of (8). It follows that the expected power of incentives is weakly lower in period 2 than in period 1 (and strictly lower, unless  $c_2^*(\theta)$  is constant in  $\theta_2$  with probability one). The same conclusion holds for  $\gamma > 1$ .

The result in the proposition then follows from continuity of the expected power of incentives in  $\gamma$ , which in turn follows from arguments analogous to those in the proof of Proposition 2. Q.E.D.

**Proof of Proposition 4.** To establish the necessity of (12) and (13), consider the perturbed effort policy  $\xi_1(\theta_1) = \xi_1^R(\theta_1) + a\nu(\theta_1)$  and  $\xi_2(\theta) = \xi_2^R(\theta) + b\omega(\theta)$  for scalars  $a$  and  $b$  and measurable functions  $\nu(\cdot)$  and  $\omega(\cdot)$ . Then differentiate the firm's profits (5) with respect to  $a$  and  $b$  respectively. A necessary condition for the proposed policy  $\xi^R$  to maximize (5) is that these derivatives, evaluated at  $a = b = 0$  vanish for all measurable functions  $\nu(\cdot)$  and  $\omega(\cdot)$ . This is true only if  $\xi^R$  satisfies (12) and (13) with probability one.

Uniqueness of  $\xi^R$  and  $c^R$ , as well as the necessity of (9), follow from the same arguments as in the proof of Proposition 1.