Neoclassical Growth Transition Dynamics with One-Sided Commitment
PRELIMINARY *

Dirk Krueger
University of Pennsylvania, CEPR and NBER

Fulin Li
University of Chicago

Harald Uhlig
University of Chicago, CEPR and NBER

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Abstract

This paper characterizes the transitionary dynamics of a continuous-time neoclassical production economy with capital accumulation in which households face idiosyncratic income risk. Insurance companies operating in perfectly competitive markets offer long-term insurance contracts and can commit to future contractual obligations, whereas households cannot. Therefore the equilibrium features imperfect insurance and a non-degenerate cross-sectional consumption distribution. When household labor productivity takes two values, one of which is zero, and the utility function is logarithmic, we show that the transition dynamics induced by unexpected positive or negative technology shocks, including the evolution of the consumption distribution, can be calculated in closed form, as long as the initial deviation from the steady state is not too large. This is in contrast to both the standard representative agent neoclassical growth model as well as Bewley (1986) style models with uninsurable idiosyncratic

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income risk. Thus the paper provides an analytically tractable alternative to the standard incomplete markets general equilibrium model developed in Aiyagari (1994) by retaining its physical structure, but substituting the assumed incomplete asset markets structure with one in which limits to consumption insurance emerge endogenously, as in the macroeconomic literature on limited commitment.

1 Introduction

Following[1] Huggett (1993) and Aiyagari (1994), a considerable literature has arisen investigating the consequences of idiosynchratic risk on individual consumption and macroeconomic outcomes, both theoretically as well as empirically. There is now considerable evidence that individual consumption smoothing is larger than what standard approaches of self-insurance via asset savings would generate. In a benchmark contribution, Blundell, Pistaferri and Preston (2008) have shown that there is a fairly low pass-through of income shocks to consumption. Using improved methods and data as well as alternative approaches, their results have been largely confirmed by the more recent literature such as Arellano, Blundell and Bonhomme (2017), Eika et al (2020), Chatterjee, Morley and Sigh (2020), Commault (2021), Neele and Lamadon (2020) and Braxton et al (2021). Thus, alternatives to the conventional self-insurance approach are needed.

This paper therefore develops a new continuous time general equilibrium neoclassical production economy with idiosyncratic income risk and insurance against these risks offered by perfectly competitive intermediaries. We assume that the intermediaries can commit to honor their future payment obligations, while agents cannot, thus limiting the degree of insurance possible. While the approach and formulation here is described by first principles, one may wish to think of the intermediaries as firms, offering partial insurance against productivity fluctuations to workers, in line with the findings in, say, Neele and Lamadon (2020).

The goal of this paper is to understand the consequences of introducing this limited insurance in an Aiyagari-type world with idiosynchratic income risk as cleanly as possible, by examining the most tractable scenario. The analysis here will thus serve as an important complement to a more quantitative and empirical, but ultimately less tractable investigation. We assume that household labor productivity takes two values, one of which is zero.

[1] It may be appropriate to also point to the PhD thesis by Uhlig (1990), which featured a model now credited to Huggett and Aiyagari, where additionally households had to make choices between risky and riskless investments. The thesis is available on microfilm.

When the utility function is logarithmic, we analytically calculate as well as analyze the steady state in Krueger and Uhlig (2021). In this paper and assuming that productivity unexpectedly changes permanently to a new level at date $t = 0$ ("MIT shock"), we show that even the transition dynamics, including the evolution of the consumption distribution, can be calculated in closed form, as long as the initial deviation from the steady state is not too large. This result is in contrast to both the standard representative agent neoclassical growth model as well as Aiyagari (1994) style models with uninsurable idiosyncratic income risk. Thus the paper provides an analytically tractable alternative and potential workhorse framework to the conventional approach.

We seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, as Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993) and Aiyagari (1994). In that model, households can trade assets to self-insure against income fluctuations, but these assets are not permitted to pay out contingent on a household's individual income realization, thereby ruling out explicit insurance against income risk.

The second branch is the broad literature on recursive contracts and endogenously incomplete markets which permits explicit insurance, but whose scope is limited by informational or contract enforcement frictions. More specifically, we incorporate dynamic insurance contracts offered by competitive financial intermediaries (as analyzed previously in Krueger and Uhlig, 2006) into a neoclassical production economy. Financial intermediaries can commit to long term financial contracts, whereas households can not. The project thereby seeks to provide the macroeconomics profession with a novel, fully micro founded, analytically tractable model of neoclassical investment, production and the cross-sectional consumption and wealth distribution, where the limits to cross-insurance are explicitly derived from first principles of contractual frictions.

Crucially, and perhaps surprisingly, analytical tractability is not limited to the stationary equilibrium, as shown in Krueger and Uhlig (2021), but extends to the entire transition path induced by unexpected (MIT) shocks to total factor productivity. This analytical tractability originates from the fact that under the assumptions made the population endogenously separates into two groups: one group with only labor income but no capital income, and a second group with no labor income but heterogeneous asset holdings and thus asset incomes. Crucially, this latter group shares the same consumption growth rate and effective saving rate, which, given log-utility is a constant. The second group then aggregates exactly, and the resulting macro economy is also characterized by a constant aggregate saving rate (as, for example in the classic Solow model or as in Moll, 2014). As Jones (2000) shows, the nonlinear ordinary differential equation characterizing the aggregate dynamics of the economy is a Bernoulli equation with a closed-firm solution; remarkably, the same is true in our economy. Given the dynamic of the aggregate capital stock, the entire transition path of the consumption distribution can also be characterized in closed form. We show that in response to a positive technology shock consumption inequality increases in the very short and converges to its original level in the long run, but the convergence need not be monotonic (i.e. inequality can undershoot the initial —and final—level, depending on parameters).

In the next section, we describe the model and define the equilibrium. Section 3 characterizes the stationary equilibrium, drawing on Krueger and Uhlig (2021). Section 4 contains the definition and the general analysis of a transition equilibrium, describing the appropriate fixed point problem and showing that much of the calculations there can already be performed analytically. A full closed form solution is then provided for the logarithmic case in section 5, when productivity changes permanently from the $t < 0$ level A^* to a new level $\tilde{A}$. We complement the analytical solutions with numerical examples in order to visually present our results. Section 6 concludes.

2 The Model

2.1 Preferences and Endowments

Time is continuous. There is a population of a continuum of infinitely lived agents of mass 1, who supply labor to the market, consume goods and sign contracts. The labor productivity z_{it} of an individual agent i at time t follows a two-state Markov process that is independent across agents. More precisely, productivity can either be high, $z_{it} = \zeta > 0$ or zero $z_{it} = 0$. Let $Z = \{0, \zeta\}$. The transition from high to low productivity occurs at rate $\xi > 0$, whereas the transition from low to high productivity occurs at rate $\nu > 0$. We assume throughout, that the productivity distribution at any date t is the stationary distribution. The stationary share of households with low and high productivity can be calculated to equal

$$(\Psi_l, \Psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu}\right) \tag{1}$$

Agents have the period CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

and discount the future at rate $\rho > 0$, so that the expected utility of a household from period t onwards is given by

$$U_t = E_t \left[\int_t^{\infty} e^{-\rho(\tau - t)} \frac{c_\tau^{1-\sigma} - 1}{1 - \sigma} d\tau \right].$$

when future period utility is discounted to period t, and where the expectation depends on the state and idiosynchratic risk of the agent[2]

There is a competitive sector of production firms which uses labor and capital to produce the final output good according to the Cobb-Douglas production function

$$A_t F(K, L) = A_t K^{\theta} L^{1-\theta}.$$

where $\theta \in (0, 1)$ denotes the capital share and $A_t > 0$ is a productivity parameter, evolving as an exogenous and non-stochastic function of time. We assume that A_t converges to a finite and strictly positive limit,

$$A_\infty = \lim_{t \to \infty} A_t, \;\; A_\infty \in (0, \infty) \tag{2}$$

The capital depreciates at a constant rate $\delta \geq 0$. Production firms seek to maximize profits, taking as given the market spot wage w_t per efficiency unit of labor and the after-deprecation rental rate or rate of return r_t per unit of capital. The scale of overall production is immaterial. We thus normalize aggregate labor supply to unity by imposing the constraint

$$\zeta = \frac{\xi + \nu}{\nu} \tag{3}$$

Similar to Krueger-Uhlig (2006, 2021), agents attempt to insure themselves against their productivity fluctuations with financial intermediaries, which offer insurance contracts against the agent-specific shocks. The financial intermediaries seek to maximize profits and are in perfect competition. Commitment is one-sided only: while the intermediary can commit to the contract for all future, agents can leave the contract at any time they please and sign up with the next intermediary. Intermediaries compete for agents, and do not have

[2]We exclude considerations of aggregate risk in this paper.

resources on their own.

The contract takes the form of a capital account k for an individual agent. One-sided commitment means, that the capital account cannot be negative, i.e. borrowing is not possible: otherwise, the agent would drop the current contract and sign up with another intermediary. The capital amount can jump, when the productivity level changes, and otherwise changes by some amount x, given the current agent-specific as well as aggregate state. Perfect competition between intermediaries means that the expected utility $U_t(k;z)$ of the agent is maximized, given the current capital amount k, agent specific productivity z and aggregate state of the economy encapsulated by the time index t. We proceed to directly state and then motivate the Hamilton-Jacobi-Bellman equation or HJB equation, in order to define the optimal contract.

Definition 1. *For $z \in Z$, wages w_t and interest rates r_t, let $\tilde{z}$ be the "other" z and let p be the transition rate $z \to \tilde{z}$. An optimal consumption insurance contract*

$$\mathcal{C}_t = \Big(U_t(k;z), c_t(k;z), x_t(k;z), \tilde{k}_t(k;z)\Big)_{k\geq 0, z\in Z}$$

solves

$$\begin{aligned} \rho U_t(k;z) &= \max_{c,x,\tilde{k}\geq 0} u(c) + \dot{U}_t(k;z) + U_t'(k;z)x + p(U_t(\tilde{k},\tilde{z}) - U_t(k;z)) && (4)\\ \textit{s.t.} \quad & c + x + p(\tilde{k} - k) = r_t k + w_t z && (5)\\ & x \geq 0 \textit{ if } k = 0 && (6)\end{aligned}$$

The Hamilton-Jacobi-Bellman equation should look familiar. For comparison, consider the standard consumption-savings problem of an agent in a riskless environment, when the agent receives a wage w_t and owns capital k, earning interest r_t. The HJB equation there reads

$$\begin{aligned} \rho U_t(k) &= \max_{c,x} u(c) + \dot{U}_t(k) + U_t'(k)x && (7)\\ \text{s.t.} \quad & c + x = r_t k + w_t && (8)\end{aligned}$$

The flow payoff $\rho U_t(k)$ of the value function $U_t(k)$ is the sum of the flow utility $u(c)$ from consuming c, the instantaneous change $\dot{U}_t(k)$ of the value function due to the passage of time and the change in the value function $U_t'(k)x$ due to the investment $\dot{k}_t = x$. Investment and consumption have to paid for, respecting the budget constraint $c + x = r_t k + w_t z$.

The agent chooses c and x so as to maximize the flow payoff $\rho U_t(k)$, given the budget constraint.

For definition 1, two features are added. First, the flow payoff $\rho U_t(k;z)$ also accounts for the expected instantaneous change in utility $p(U_t(\tilde{k},\tilde{z}) - U_t(k;z))$ due to a possible change in productivity from z to $\tilde{z}$. The change in the capital stock has to be paid for and thus accounted for in the budget constraint, using the actuarily fair amount $p(\tilde{k} - k)$. Second, the lack of commitment by the agent or borrowing constraint is incorporated per the restriction that $\tilde{k} \geq 0$ as well as $x \geq 0$, when $k = 0$.

2.2 Equilibrium

Imposing suitable conditions, we will focus entirely on equilibria, such that high productivity agents never wish to hold capital. The only reason for holding capital is thus to finance the consumption stream of zero productivity agents. High productivity agents pay insurance premia to obtain a stock of capital, should the transition to zero productivity occur. Thus, all high productivity agents are identical: we do not need to keep track of their history. Low productivity agents are distinguished by the length of time $\tau \geq 0$ elapsed, since the transition from high to low productivity occurred.

The distribution over these agent types is easy to characterize. The total mass of high and low productivity agents is given in equation (1). The density for low productivity agents is given by

$$\psi_l(\tau) = \frac{\xi\nu}{\xi+\nu} e^{-\nu\tau},\ \tau \geq 0 \tag{9}$$

whjch integrates to the total mass $\Psi_l = \xi/(\xi+\nu)$ of low productivity agents. Low productivity agents hold capital $k_{s,t}$ depending on the date t and the time $s = t - \tau$ of the transition to low productivity. Thus, rather than keep track of the joint state distribution across capital and productivity states $(k;z)$, it is more convenient to keep track of the capital holding $k_{s,t}$ as a function of the transition time s and the calendar time t. Time derivatives are always with respect to calendar time.

Definition 2. *An equilibrium consists of consumption insurance contracts $\mathcal{C}_t$, equilibrium wages w_t, interest rates r_t, aggregate capital K_t and capital holdings of low productivity agents $(k_{s,t})_{s \leq t}$, as functions of time $t \in (-\infty, \infty)$, such that*

1. *Given the evolution of w_t and r_t, the consumption insurance contracts $\mathcal{C}_t$ are optimal in the sense of definition 1.*

2. *The contracts $\mathcal{C}_t$ have the "only low productivity agents hold capital" property that $\tilde{k}_t(k;0) = 0$ for all $k = k_{t,\tau}$, $\tau \geq 0$ as well as $x_t(0;\zeta) = 0$.*

3. *The capital holdings of low productivity agents are consistent with the contracts $\mathcal{C}_t$, i.e.*

$$k_{t,t} = \tilde{k}_t(0;\zeta) \tag{10}$$

$$\dot{k}_{s,t} = x_t(k_{s,t};0) \tag{11}$$

where $\dot{k}_{s,t} = \partial k_{s,t}/\partial t$.

4. *The interest rates and wages (r_t, w_t) satisfy*

$$r_t = A_t F_K(K_t, 1) - \delta \tag{12}$$

$$w_t = A_t F_L(K_t, 1) \tag{13}$$

5. *The goods market clears*

$$\int_0^\infty c_t(k_{t-\tau,t};0)\psi_l(\tau)d\tau + \frac{\nu}{\xi+\nu}c_t(0;\zeta) + \delta K_t = A_t F(K_t, 1). \tag{14}$$

6. *The capital market clears*

$$\int_0^\infty k_{t-\tau,t}\psi_l(\tau)d\tau = K_t \tag{15}$$

3 Stationary Equilibrium

The existence and properties of stationary equilibria for a constant $A_t \equiv A^*$ are discussed in Krueger-Uhlig (2021). In particular, suppose that agents have log preferences, $\sigma = 1$. Impose the

Assumption 1. *Let the exogenous parameters of the model satisfy $\theta, \nu, \xi, \rho > 0$ and*

$$\frac{\theta}{(1-\theta)(\rho+\delta)} < \frac{\xi}{\nu(\rho+\nu+\xi)} \tag{16}$$

Define

$$\alpha = \frac{\nu+\rho}{\nu+\rho+\xi}\zeta \tag{17}$$

The result in Krueger-Uhlig (2021) can be restated as

Proposition 1. *[Krueger-Uhlig (2021)] Let assumption 1 be satisfied. Then there exists a unique stationary equilibrium. The equilibrium features partial insurance, i.e. consumption of the high productivity agents is c_h and consumption of the low-productivity agents drifts downwards at rate $r^* - \rho < 0$. The unique equilibrium interest rate is given by*

$$r^* = \frac{\theta(\nu + \rho + \xi)(\nu + \rho) - \xi\delta(1 - \theta)}{\theta(\nu + \rho) + \xi} < \rho \tag{18}$$

The equilibrium capital stock is

$$K^* = \left(\frac{\theta A^*}{r^* + \delta}\right)^{\frac{1}{1-\theta}} \tag{19}$$

and the equilibrium wage is

$$w^* = (1 - \theta)A^* \left(\frac{\theta A^*}{r^* + \delta}\right)^{\frac{\theta}{1-\theta}} \tag{20}$$

The stationary consumption distribution has a mass point at

$$c_h^* = \alpha w^* \tag{21}$$

for the mass $\nu/(\nu + \xi)$ of high-productivity agents and

$$k_\tau^* = e^{-(\rho - r^*)\tau} \frac{c_h^*}{\nu + \rho} \tag{22}$$

$$c_\tau^* = e^{-(\rho - r^*)\tau} c_h^* \tag{23}$$

for the low-productivity agents as a function of τ since their transition to low productivity, where $k_\tau^* = k_{t-\tau,t}$ and $c_\tau^* = c(k_{t-\tau,t})$ is independent of t.

Note that k_τ^* is the net present value of the future zero-income consumptions $c_{\tau+s}^*$, taking into account the rate ν of switching out of the zero income state,

$$k_\tau^* = \int_{s=0}^{\infty} e^{-(\nu + r^*)s} c_{\tau+s}^* ds \tag{24}$$

Figure 1 illustrates the insurance arrangement of an agent, with productivity z_t and thus labor income $y_t = w^* z_t$ switching at Poisson dates between high and zero productivity resp high and zero labor income. In the high income state and due to assumption 1, the agent holds no capital. He consumes less than his income, using the difference to make insurance payments against the possibility of a switch to low productivity. When the switch occurs, the agent receives a stock of capital as insurance payout and uses that to finance his consumption stream during the zero productivity phase.

4 Transition Equilibrium

We now seek to characterize the full dynamic equilibrium when $A_t = A^*$ and the economy starts in the stationary equilibrium characterized in section 3 for $t < 0$, and then A_t becomes a function of time for $t \geq 0$, inducing an equilibrium transition towards a new stationary equilibrium. We assume that this change in productivity happens as a complete surprise ("MIT shock"), i.e. the contracts signed for $t < 0$ have not allowed for that contingency.

Thus, all equilibrium variables satisfy the equilibrium definition for the stationary equilibrium $A_t \equiv A^*$ for $t \leq 0$, thereby determining the initial conditions at date $t = 0$. The capital stock does not change at date 0. Correspondingly, we also assume that the agent-specific capital holdings $k_{s,0}$ of low productivity agents for $s \leq 0$ and at date $t = 0$ are those delivered by the steady state equilibrium for $A_t \equiv A^*$.

Figure 2 illustrates the transition at date 0 and the insurance arrangements. While the capital held by zero productivity agent remains unchanged, a different path for consumption emerges, due to changed aggregate dynamics in r_t and w_t and the resulting income process $y_t = w_t z_t$.

Definition 3. *A transition equilibrium consists of consumption insurance contracts $\mathcal{C}_t$, equilibrium wages w_t, interest rates r_t, capital K_t and capital holdings of low productivity agents $(k_{s,t})_{s \leq t}$, satisfying the equilibrium conditions for $t \geq 0$, where K_0 and $(k_{s,0})_{s \leq 0}$ coincide with the equilibrium values of a stationary equilibrium for $A_t \equiv A^*$.*

We assume that the transition starts from a stationary equilibrium $K^*, r^*, w^*, \mathcal{C}^*, \Psi^*$ for $A_t \equiv A^*$ and that such an equilibrium exists, see Krueger-Uhlig (2021) and section 3. A transition equilibrium is now characterized via the following fixed point problem.

1. Conjecture a path for aggregate capital K_t along the transition, given the initial condition $K_0 = K^*$. Calculate r_t and w_t, using the first-order condition of production firms, i.e. capital and labor demand. (Section 4.1)

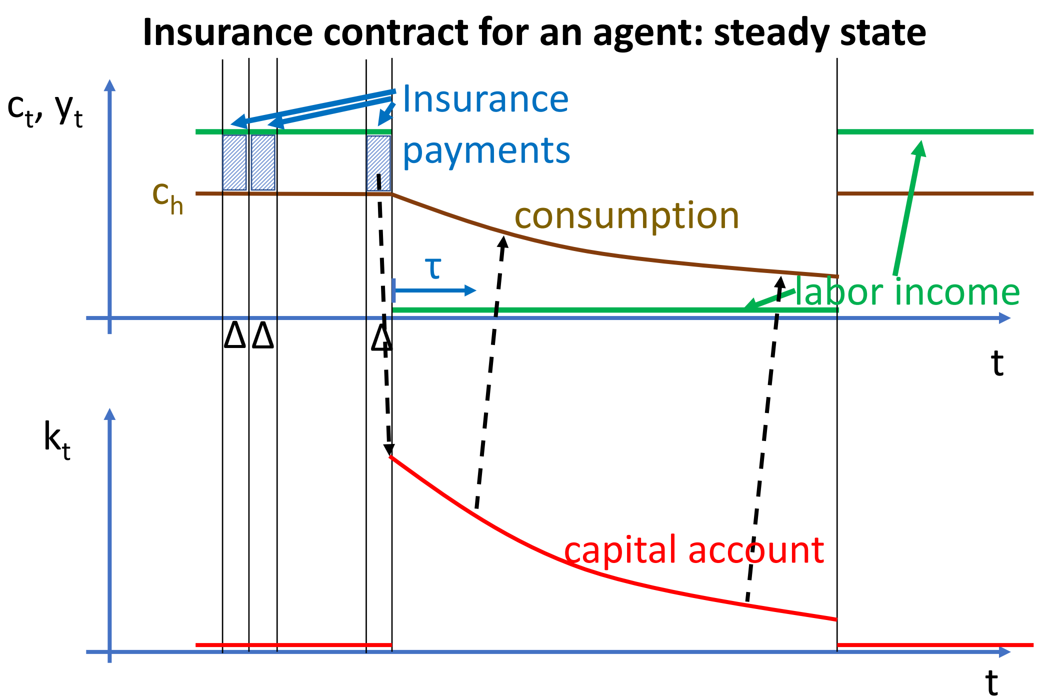


Figure 1: Insurance contract in the steady state equilibrium. In the high income state, the agent holds no capital and consumes less than his income, using the difference to pay insurance against a productivity change. When the productivity state changes to zero, the agent receives a stock of capital as insurance payouts, running it down, while productivity and thus wage income is zero. During that phase, the agent also receives and consume insurance payouts against a return to the high productivity state, backed by his capital holding as collateral. When productivity switches to the high state again, the capital holding is returned to the intermediary. The instantaneous nature of these insurance arrangements imply that the agent is indifferent between available intermediaries and can switch at any time, given his insurance premia in the high income phase and his capital holdings in the low income phase.

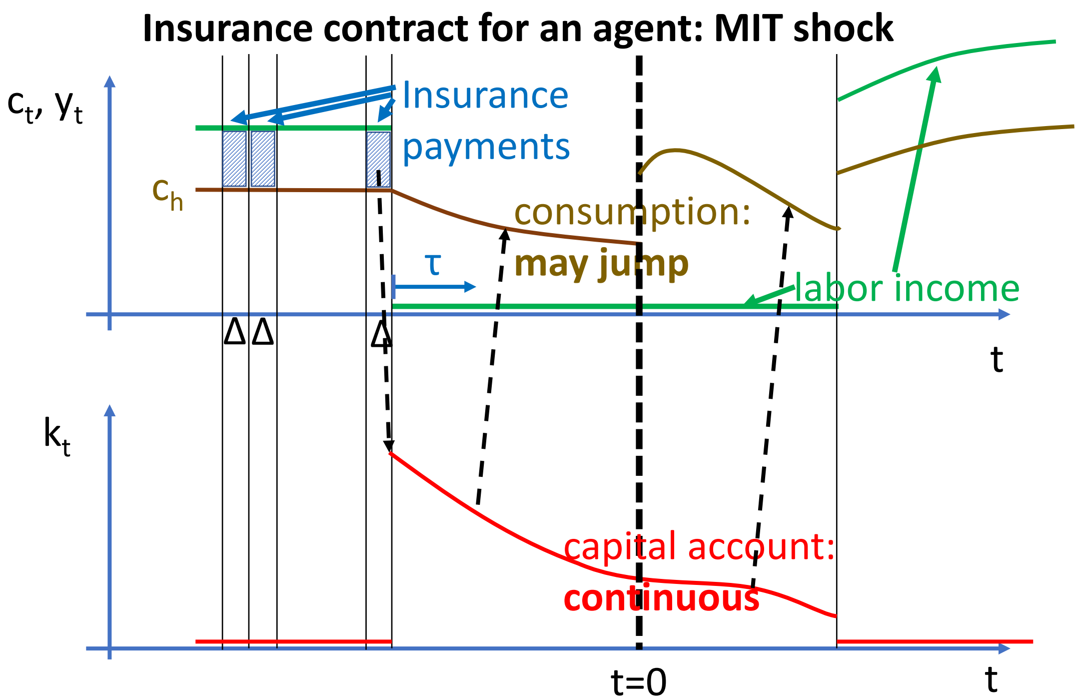


Figure 2: Insurance contract at the transition $t = 0$. Low productivity agent keep their capital. However, since returns r_t and wages w_t have changed, a different path for consumption can now be financed, given this capital holding.

2. Characterize the optimal contracts $\mathcal{C}_t$, given the paths for r_t and w_t. (Section 4.2)

3. Compute the path of aggregate capital supply K_t^S by aggregating the capital holdings across individual households. (Section 4.3)

4. Check whether the path of aggregate capital supply K_t^S matches the conjectured path of aggregate capital stock K_t in the first step.

In this section we will proceed with a general CRRA utility function, since, conceptually, nothing depends on the period utility function being logarithmic. In the next section we will specialize to the log-case to obtain closed-form solutions.

4.1 Conjecture a path for capital.

Conjecture a path for capital $(K_t)_{t\geq 0}$. With that and from the first-order condition of production firms, i.e. from capital demand, calculate the path for interest rates and wages,

$$\begin{aligned} r_t &= \theta A_t K_t^{\theta-1} - \delta && (25) \\ w_t &= (1-\theta) A_t K_t^{\theta} && (26) \end{aligned}$$

Since the interest rate in stationary equilibrium does not depend on the value of aggregate productivity A (as in the standard neoclassical growth model), the interest rate will converge to the original steady state value r^*,

$$\lim_{t\to\infty} r_t = r^* \tag{27}$$

This together with (25) implies that capital K_t must converge to

$$K_\infty = \left(\frac{\theta A_\infty}{r^* + \delta}\right)^{\frac{1}{1-\theta}} \tag{28}$$

We therefore see, that the conjecture for a capital path $(K_t)_{t\geq 0}$ is constrained by the two boundary conditions of $K_0 = K^*$, the original steady state as inititial condition for $t = 0$, and $\lim_{t\to\infty} K_t = K_\infty$.

4.2 Characterize the optimal contracts $\mathcal{C}_t$

Equipped with a conjectured path of wages and interest rates, the only aggregate variables relevant for the dynamic insurance problem at the household level, we can now characterize the optimal contract by deriving conditions for the evolution of consumption and capital over time and as a function of productivity. Define

$$g_t = \frac{\rho - r_t}{\sigma} \tag{29}$$

As might be expected from the standard consumption-savings problem with CRRA utility, g_t will turn out to be the negative of the consumption growth rate, if capital holdings are strictly positive. We define

$$D_t \equiv \int_t^{+\infty} e^{-\int_t^s (r_u + \nu + g_u) du} ds \tag{30}$$

which will help to calculate the net present value of a stream of future consumption. Given the contract $\mathcal{C}_t$, we define the implied time derivative of consumption as[3]

$$\dot{c}_t(k; z) \equiv \frac{\partial c_t(k; z)}{\partial t} + \frac{\partial c_t(k; z)}{\partial k} x_t(k; z) \tag{31}$$

Lemma 1 (The contract $\mathcal{C}_t$ for $z = 0$ and $k > 0$.)**.** *For $k > 0$, the optimal contract of definition 1 implies the consumption dynamics*

$$\frac{\dot{c}_t(k; z)}{c_t(k; z)} = -g_t \tag{32}$$

Furthermore, if $z = 0$ and $\tilde{k}_t(k; 0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, then

$$c_t(k; 0) = \frac{k}{D_t} \tag{33}$$

$$x_t(k; 0) = \left(r_t + \nu - \frac{1}{D_t} \right) k \tag{34}$$

for all $k \leq \bar{\bar{k}}$ and some suitably chosen $\bar{\bar{k}}$.

[3] As a heuristic for this definition of the time derivative, suppose that productivity remains constant at z for some interval of time. In that case, note that $\dot{k}_t = x_t(k; z)$ and that consumption evolves as $c(t) = c_t(k_t; z)$ as a function of time only. Taking the derivative with respect to time yields the expression here.

The proof is in appendix A.1. We now use this result to characterize the dynamics of consumption for individuals with currently high productivity. To do so, let us make

Assumption 2. *Suppose the aggregate wage and interest rate satisfies the following condition, for all $t \geq 0$:*

$$0 < \frac{\dot{w}_t}{w_t} + g_t - \frac{\xi \dot{D}_t}{\zeta w_t} \tag{35}$$

Lemma 2 (The contract $\mathcal{C}_t$ for $z = \zeta$ and $k = 0$)**.** *Let assumption 2 be satisfied. Then the optimal contract of definition 1 implies the dynamics of consumption and investment for individuals with currently high productivity as*

$$c_t(0;\zeta) = \frac{w_t \zeta}{1 + \xi D_t} \tag{36}$$

$$x_t(0;\zeta) = 0 \tag{37}$$

Furthermore,

$$\tilde{k}_t(0;\zeta) = \frac{w_t \zeta}{1 + \xi D_t} D_t \tag{38}$$

The proof is in appendix A.2. The term ξD_t in the numerator of the right hand side of (36) is the insurance premium to obtain the capital stock $\tilde{k}_t(0;\zeta)$ in case of a transition to the zero productivity state and to assure desirable continuity of consumption via (33), if so. Indeed, the equality

$$\xi \tilde{k}_t(0;\zeta) = w_t \zeta - c_t(0;\zeta)$$

shows that this is an actuarily fair contract. Note that (36) implies

$$\frac{\dot{c}_t(0;\zeta)}{c_t(0;\zeta)} = \frac{\dot{w}_t}{w_t} - \frac{\xi \dot{D}_t}{\zeta w_t} \tag{39}$$

Equation (39) rationalizes the condition (35). If consumption could be chosen in an unconstrained fashion, then we would obtain (32). With (35), consumption would grow more slowly than the right hand side of (39), but this can now only be accomplished per borrowing against future wages and choosing $x < 0$, subject to making the consumption-smoothing insurance payments against the transition to zero productivity. But this is ruled out by the borrowing constraint (6). Put differently, condition (35) assures, that the high productivity agent has no desire to accumulate capital.

For ease of notation, let $c_{s,t} = c_t(k_{s,t}; 0)$ denote the consumption of a zero productivity agent at date t, who switched from high to zero productivity at date $s \leq t$, and thus holds capital $k_{s,t}$. Per equation (33), capital holdings are proportional to consumption for low productivity agents,

$$k_{s,t} = D_t c_{s,t} \tag{40}$$

Equations (36) and (38) imply that

$$\begin{aligned} c_{s,s} &= c_s(0;\zeta) = \frac{w_s \zeta}{1 + \xi D_s} && (41) \\ k_{s,s} &= c_s(0;\zeta) = \frac{w_s \zeta}{1 + \xi D_s} D_s && (42) \end{aligned}$$

due to consumption smoothing. For zero productivity agents, the consumption growth equation (32) or

$$\frac{\dot{c}_{s,t}}{c_{s,t}} = -\frac{\rho - r_t}{\sigma} = -g_t \tag{43}$$

holds except for the economy-wide "MIT-shock" date $t = 0$ of transiting to the productivity evolution A_t. If an agent last switched from high to zero income after that transition date, i.e. if $s > 0$, then equation (A.8) characterizes the consumption evolution since that date. If the switch last happened at some date $s \leq 0$, the zero income agent will have started at some steady state capital k^*_{-s}, characterized by (22) in the log-utility scenario. More generally, using the results above applied to the steady state and with

$$g^* = \frac{\rho - r^*}{\sigma} \tag{44}$$

and

$$D^* = \frac{1}{\nu + r^* + g^*} \tag{45}$$

we have

$$\begin{aligned} c^*_h &= \frac{w^* \zeta}{1 + \xi D^*} && (46) \\ c^*_\tau &= e^{-g^* \tau} c^*_h && (47) \\ k^*_\tau &= D^* e^{-g^* \tau} c^*_h && (48) \end{aligned}$$

These two cases for s yield the consumption dynamics of zero income agents.

Lemma 3 (Consumption dynamics for zero income agents)**.** *Consider the consumption $c_{s,t}$ of a zero income agent at time t, who last switched from $z = \zeta$ to $z = 0$ at date $s \leq t$.*

1. If $s > 0$, then

$$c_{s,t} = e^{-\int_s^t g(u)du} \frac{w_s \zeta}{1 + \xi D_s} \tag{49}$$

2. If $s \leq 0$, then

$$c_{s,t} = e^{-\int_0^t g_u du} \frac{k^*_{-s}}{D_0} \tag{50}$$

Equation (49) can be rewritten With equation (42) as

$$c_{s,t} = e^{-\int_s^t g(u)du} \frac{k_{s;s}}{D_s} \tag{51}$$

or, more generally, as

$$c_{s,t} = e^{-\int_q^t g(u)du} \frac{k_{s;q}}{D_q} \tag{52}$$

for any $s \leq q \leq t$. Comparing (50) and (52) shows, that consumption for agents with $s < t$ will jump, if and only if $D_0 \neq D^*$: the change in the path of future interest rates may induce the agent to reduce or to increase current consumption, compared to the steady state and given the same budget or net present value at date $t = 0$.

4.3 Compute the path of aggregate capital supply

To compute the aggregate capital supply K_t at time t, we aggregate the capital holdings of low productivity agents,

$$K_t = \int_{-\infty}^{t} k_{s,t} \psi_l(t-s) ds \tag{53}$$

Lemma 4 (Dynamics of aggregate capital supply)**.** *Capital supply evolves according to*

$$\dot{K}_t = \left(\frac{\xi D_t}{1 + \xi D_t} (1 - \theta) + \theta \right) A_t K_t^\theta - \left(\delta + \frac{1}{D_t} \right) K_t \tag{54}$$

where D_t was defined in equation (30) *and is reproduced here:*

$$D_t \equiv \int_t^{+\infty} e^{-\int_t^s (r_u + \nu + g_u) du} ds \tag{55}$$

The proof is in appendix A.3. Given initial capital stock $K_0 = K^*$, we can use the capital dynamics in equation (54) to solve for the path of aggregate capital supply. Lemma

5 summarizes the results, see appendix A.4 for the proof.

Lemma 5 (Aggregate capital supply)**.** *Suppose capital evolves according to equation* (54) *for any time $t \geq 0$, given an initial condition K_0. Then the aggregate capital supply at any time $t \geq 0$ takes the following form*

$$K_t^S = \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)\int_s^t b_u du} a_s ds\right)^{\frac{1}{1-\theta}} \tag{56}$$

where

$$a_t \equiv \left(\frac{\xi D_t}{1+\xi D_t}(1-\theta)+\theta\right) A_t \tag{57}$$

$$b_t \equiv \delta + \frac{1}{D_t} \tag{58}$$

and D_t is defined in equation (55)*.*

5 Logarithmic Utility: $\sigma = 1$

In this section we summarize the key simplifications of the dynamic consumption contract if, in addition to the assumption on the idiosyncratic productivity process, we also assume that the period utility function is logarithmic. With log utility, $\sigma = 1$, the key simplification, due to the usual cancellation of income and substitution effects on consumption of interest rate changes, is that $g_t = \rho - r_t$, $r_t + g_t = \rho$. Then the discount function in equation (55) simplifies to a constant, rather than depend on future interest rates:

$$D_t \equiv \int_t^{+\infty} e^{-\int_t^x (r_u+g_u+\nu)du} dx = \frac{1}{\rho+\nu} \tag{59}$$

This in turn dramatically simplifies the equilibrium consumption allocation and the associated aggregate dynamics of capital, which can be solved in closed form.

Equations of (33) and (34) of Lemma 1 become the particularly simple relationships

$$c_t(k;0) = (\rho+\nu)k \tag{60}$$

$$x_t(k;0) = -(\rho - r_t)\, k \tag{61}$$

Intuitively, with log utility and without wage income, the fraction of capital spent on consumption does not depend on future interest rates: a result well familiar from many other sources. As long as Assumption 2 is satisfied and as a direct consequence of Lemmas 2 and 3, the consumption allocation with log-utility is given by

$$\frac{c_{h,t}}{w_t} = \alpha = \frac{c_h^*}{w^*} \tag{62}$$

$$c_{s,t} = \begin{cases} e^{-\int_s^t g_u du} c_{h,s} & \text{if } s > 0 \\ e^{-\int_0^t g_u du} (\rho + \nu) k_{-s}^* & \text{if } s \leq 0 \end{cases} \tag{63}$$

for all s, t, where α is defined in (17) and k_τ^* is defined in (22) and where we recall that the unconstrained growth rate of consumption is given by $g_t = \rho - r_t$. Given the discussion following (52), we see that consumption will not jump at $t = 0$ for zero income agents with $s < 0$, since $D_0 = D^* = 1/(\rho + \nu)$. Thus, one can summarize (63) as

$$c_{s,t} = e^{-\int_s^t g_u du} c_{h,s} \tag{64}$$

where it is understood that for $t < 0$, i.e. prior to the MIT shock the interest rate is equal to the steady state interest rate, $r_t = r^*$.

The remarkable part of the consumption allocation is that the entry level of consumption from which consumption drifts down upon receiving the negative shock, $c_{h,t}$, when normalized by the wage, is the same as in the initial steady state, despite the fact that the subsequent consumption path drifts down at a different (and time-varying) rate (see equation). However, since the rates at which future consumption is discounted also changes, with log-utility the present discounted value of this altered consumption stream remains the same, and thus the entry level of consumption (relative to the wage) does not change along the transition path.

Under Assumptions 1 and 2, Lemma 4 gives the aggregate law of motion for capital as

$$\dot{K}_t = \left(\frac{\xi}{\rho + \nu + \xi} (1 - \theta) + \theta \right) A_t K_t^\theta - (\delta + \rho + \nu) K_t$$

From lemma 5, the entire equilibrium time path for the capital stock is then given in closed form as

$$K_t = \left(e^{-(1-\theta)(\delta+\rho+\nu)t} (K_0)^{1-\theta} + (1 - \theta) \int_0^t e^{-(1-\theta)(\delta+\rho+\nu)(t-s)} a_s ds \right)^{\frac{1}{1-\theta}} \tag{65}$$

for all time $t \geq 0$, where, for any time s,

$$a_s = \left(\frac{(1-\theta)\,\xi}{\rho+\nu+\xi} + \theta \right) A_s.$$

In contrast to the general case, where the integral in equation (65) involved future interest rates (and thus equilibrium entities), with log-utility the right-hand side of (65) is exclusively a function of exogenous parameters and the exogenous time path for total factor productivity A_t. The explicit solution in (65) in principle applies to any productivity path, but the requirement that Assumption 2 be satisfied imposes restrictions on the path for which (65) is a valid characterization of the equilibrium transition path for capital.

The time paths of all other aggregate variables, such as interest rates, wages and aggregate consumption directly follow from that of the aggregate capital stock, as in the standard neoclassical growth model. For all $t \geq 0$:

$$\begin{aligned} r_t &= \theta A_t K_t^{\theta-1} - \delta \\ w_t &= (1-\theta)\, A_t K_t^{\theta} \\ C_t &= A_t K_t^{\theta} - \delta K_t - \dot{K}_t \\ &= (1-\theta)\, \frac{\rho+\nu}{\rho+\nu+\xi} A_t K_t^{\theta} + (\rho+\nu)\, K_t \end{aligned}$$

5.1 Permanent Increase in A

In this subsection and the next, we analytically characterize the aggregate and distributional consequences of a permanent shock to total factor productivity in closed form. We do so for both a positive shock in here, followed by the analysis of a permanent *decline* in productivity in subsection 5.2). The purpose of doing so is threefold: first, for these cases we can give straightforward sufficient conditions for Assumption 2 to be satisfied, second, we can display graphically the transition path characterized analytically and third, we can characterize, again analytically, the impact of permanent changes of productivity on consumption inequality.

Assume that at $t = 0$, productivity increases from A^* to $\tilde{A}$ and it remains $\tilde{A}$ thereafter. Hence,

$$A_t = \tilde{A}, \forall t \geq 0$$

We maintain that Assumption 1 be satisfied, guaranteeing the existence of a unique stationary equilibrium with partial insurance from which the transition path starts. This assumption will also guarantee that the interest rate is monotonically decreasing along the transition path induced by a permanent increase in productivity from A^* to $\tilde{A}$.

We can now replace Assumption 2 with Assumption 3 stated purely in terms of exogenous parameters; it requires that the permanent productivity shock cannot be too large. Define

$$\bar{A} = A^* \left(1 + \frac{\nu\,(\rho+\delta)}{\theta\,(\rho+\nu+\delta)} \left(1 + \frac{\xi}{\rho+\nu}\right) \left(\frac{\xi}{\nu\,(\rho+\nu+\xi)} - \frac{\theta}{(1-\theta)\,(\rho+\delta)}\right)\right) \tag{66}$$

Assumption 3. *[The permanent increase in productivity is not too large] The new permanent level $\tilde{A}$ satisfies $\tilde{A} < \bar{A}$.*

Under Assumptions 1 and 3, we can fully characterize the transitional dynamics in closed form. Proposition 2 summarizes the results. See Appendix A.5 for the proof. As a side note, the proof of Proposition 2 applies the results in Lemma 6, which is proved in Appendix A.6.

Proposition 2 (Transitional dynamics under log utility after a permanent increase in A)**.** *Suppose the households have log utility and a permanent shock raises productivity from A^* to $\tilde{A}$. Further impose Assumption 1 and 3 Then for all times $t \geq 0$,*

1. *The aggregate capital stock is given by*

$$K_t = \left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{\frac{1}{1-\theta}}$$

where

$$\begin{aligned} a &= \left(\frac{\xi}{\rho+\nu+\xi}\,(1-\theta) + \theta\right)\tilde{A} \\ b &= \delta+\rho+\nu \\ \frac{a}{b} &> (K^*)^{1-\theta} \end{aligned}$$

The equilibrium interest rate is

$$r_t = \theta\tilde{A}\left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{-1} - \delta$$

and the equilibrium wage is

$$w_t \;=\; (1-\theta)\,\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}}$$

2. *The aggregate capital stock is strictly increasing over time. The equilibrium interest rate is strictly decreasing over time, and the equilibrium wage is strictly increasing over time. They satisfy assumption 2.*

Lemma 6. *Suppose the households have log utility, and a permanent shock raises productivity from A^* to $\tilde{A}$. Further impose Assumptions 1 and 3. Then*

$$\frac{a}{b} \;>\; (K^*)^{1-\theta}$$

where a and b are defined in Lemma 2.

Consumption Inequality Our model is tractable enough to characterize the evolution of consumption inequality after a shock to poroductivity in closed form. Consider first consumption of high income households. From equation (62)

$$c_{h,t} = \alpha w_t$$

where α is given in equation (17). Hence, the highest income household's consumption normalized by wage, remains constant over time. This also means that the highest consumption level (in the consumption distribution) increases over time due to wage increase.

Now we turn to the low income households. For a type 2 household, define the time elapsed since last high income as $\tau = t - s$, so $s = t - \tau$.

For a type 2 household with parameter τ, we want to characterize the gap between his consumption and the highest consumption level at any time t, which is a measure of consumption inequality. Lemma 7 shows that, for a type 2 household with parameter τ, the consumption gap first shrinks then widens over time. Under the new stationary equilibrium, consumption gap is the same as that under the old stationary equilibrium.

Lemma 7. *Suppose the households have log utility, and a permanent shock raises productivity from A^* to $\tilde{A}$. Further impose Assumptions 1 and 3.*

Define the "consumption gap" as the log consumption ratio between the high income household and a low income household with τ. Then at any time t, the consumption gap *can be expressed as*

$$-\log\left(\frac{c_{t-\tau,t}}{c_{h,t}}\right) = \underbrace{-\log\left(\frac{w_{t-\tau}}{w_t}\right)}_{\textit{wage gap}} + \underbrace{\int_{t-\tau}^{t} g_u du}_{\textit{discounting gap}}$$

where we define the first term as the "wage gap", and the second term as the "discounting gap".

The consumption gap, wage gap and discounting gap have the following properties:

1. *At $t = 0$ upon the aggregate technology shock, the wage gap widens (discontinuously) while the discounting gap shrinks (continuously), so the consumption gap widens (discontinuously) upon the shock.*

2. *For $t \in (0, \tau)$, the wage gap widens (continuously) while the discounting gap shrinks (continuously). The latter effect dominates, so the consumption gap shrinks (continuously).*

3. *At $t = \tau$, the wage gap shrinks (discontinuously) and the discounting gap shrinks (continuously), so the consumption gap shrinks (discontinuously).*

4. *For $t > \tau$, the wage gap shrinks (continuously) while the discounting gap widens (continuously). The latter effect dominates, so consumption gap widens (continuously).*

5. *When the economy reaches the new stationary equilibrium, the wage gap, discounting gap and consumption gap all revert back to their original levels at the old stationary equilibrium.*

See Appendix A.7 for the proof. Appendix B.1 characterizes the components of the consumption gap in closed from.

What is the inution for this result? At $t = 0$ upon the shock, low income households do not benefit from the wage increase, so the wage gap widens sharply. Within the small time interval dt, the change in the discounting gap is negligible relative to the discontinuous change in the wage gap. Hence, the consumption gap widens. And it is exactly because the high income household obtains a new contract which reflects the wage jump, while a low income household's new contract does not reflect the higher wage until the next time he switches to high income.

During the time period $t \in (0, \tau)$, for a low income household with τ, his consumption still drifts down from an old level $c_h(t-\tau) = c_h^*$, which does not reflect the wage increase. Hence, the wage gap widens. However, he now enjoys the benefit of the higher interest rate, because his consumption now drifts down at a lower rate, which means the discounting gap shrinks. It turns out that the discounting effect dominates, and thus the consumption gap shrinks.

At time $t = \tau$, the household with τ is the household who last had high income at time 0. This means that his consumption now drifts down from a high level $c_h(0) > c_h^*$, which reflects the wage jump on impact. So the wage gap shrinks discontinuously. Moreover, within the small time interval dt, the change in the discounting gap is negligible relative to the wage jump. Hence, the consumption gaps shrinks.

For $t > \tau$, the consumption gap, wage gap and discounting gap revert back to their steady state levels.

- Let's first consider the wage gap. The "current" equilibrium wage w_t is always higher than the wage at the point when the low income household signed the contract, i.e. $w_t > w(t-\tau)$. Let call $w(t-\tau)$ the "old" wage. A household's consumption depends on the wage at the point when the contract was signed, so the wage gap $\frac{w(t-\tau)}{w_t}$ is related to the consumption inequality. As time goes by, the "current" wage w_t increases, and the "old" wage $w(t-\tau)$ also increases. So the wage gap will depend on the concavity of the wage path. It turns out that the wage path is concave in time, which implies that as time goes by, the "old" wage and the "new" wage are getting closer to each other, and thus the wage gap shrinks.

  To put it differently, as the economy approaches the new steady state, every household will hold a contract based on a wage that is closer to the wage under the new steady state. And this is due to the concavity of the wage path.

- Next, we consider the discounting gap. The discounting gap comes from a unique feature of the consumption insurance contract – for low income household, consumption drifts down over time. As time goes by, interest rate decreases, so the low income household with τ discounts its consumption more heavily from the initial high consumption level (when the contract was signed). This implies that the discounting gap widens.

- It turns out that the discounting effect dominates, so the consumption gap widens.

As a final remark, we see that along the transitional path, the discounting effects dominates the wage effect whenever the two effects work in opposite directions.

Mapping between τ, quantiles in the population, quantiles in the consumption distribution, and consumption gap Let ι denotes the consumption ratio of a low income household to a high income household, P denote the quantile in the population of households ranked by consumption level, G denote the quantile in the consumption distribution.

Table 1: Notations

Symbol	Definition
τ	time since the low income household last had high income
ι	consumption ratio of a low income household to a high income household
P	quantile in the population ranked by consumption level
G	quantile in the consumption distribution

Mapping between τ and P We first establish a one-to-one mapping between τ and the quantiles in the population ranked from lowest to highest consumption. Equations (62) and (63) characterizing the optimal allocations imply that, for $\forall\tau,\tau'$ such that $0<\tau'<\tau$,

$$\frac{c_{t-\tau,t}}{c_{t-\tau',t}} = \frac{w\left(t-\tau\right)}{w\left(t-\tau'\right)}e^{-\int_{t-\tau}^{t-\tau'}g(u)du}<1$$

Hence, τ fully characterizes the household's rank in the consumption distribution. In particular, a lower τ corresponds to a higher rank (a higher consumption level). This is because the household with a lower τ has signed a contract more recently, which allows him to enjoy the benefit of both a higher wage and less discounting.

At any time t, we compute the fraction of households with lower consumption than the household who last had high income at time $t-\tau$. Denote it as $P\left(\tau\right)$. Then

$$P\left(\tau\right) = \int_{\tau}^{+\infty}\psi_l(x)dx=\int_{\tau}^{+\infty}\frac{\xi\nu}{\xi+\nu}e^{-\nu x}dx=\frac{\xi}{\xi+\nu}e^{-\nu\tau}$$

And $P\left(\tau\right)$ is the one-to-one mapping from τ to the quantiles in the population (ranked by consumption).

We can also do it the other way around – establish the mapping from the quantiles in the population to τ. Given a P, i.e. the share of households on the lower end of the consumption distribution,we can find the τ such that P fraction of the households have lower consumption than the household who last had high income at time $t-\tau$.

$$\tau(P) = \frac{1}{\nu}\log\left(\frac{\xi}{(\xi+\nu)P}\right)$$

Mapping from ι to P Given an ι (the consumption ratio between a low income household and a high income household,) we can compute the fraction of households whose consumption ratio is below ι.

For $\iota < 1$,

$$\begin{aligned}
P(\iota,t) &= Pr\left(\frac{c_{t-\tau,t}}{c_{t;h}} \leq \iota\right) = Pr\left(-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) \geq -\log\iota\right) \\
&= \int_{\underline{\tau}}^{+\infty}\psi_l(x)dx = \int_{\underline{\tau}}^{+\infty}\frac{\xi\nu}{\xi+\nu}e^{-\nu x}dx \\
&= \frac{\xi}{\xi+\nu}e^{-\nu\underline{\tau}(\iota,t)}
\end{aligned}$$

where $\underline{\tau}(\iota,t)$ is the solution to the following equation

$$\frac{c_{t-\underline{\tau},t}}{c_{t;h}} = \iota$$

For $\iota = 1$,

$$P(1,t) = 1$$

Mapping from P to G Given P (the cumulative share of households from lowest to highest consumption), we can compute the cumulative consumption share of these households.

$$\begin{aligned}
G(P,t) &= \frac{1}{C_t}\int_{\tau(P)}^{+\infty}c_{t-x,t}\psi_l(x)dx \\
&= \frac{1}{C_t}\int_{\tau(P)}^{+\infty}c_h(t-x)\,e^{-\int_{t-x}^{t}g(u)du}\frac{\xi\nu}{\xi+\nu}e^{-\nu x}dx \\
&= \frac{1}{C_t}\frac{\xi\nu}{\xi+\nu}\frac{z}{1+\frac{\xi}{\rho+\nu}}\int_{\tau(P)}^{+\infty}w(t-x)\,e^{-\int_{t-x}^{t}g(u)du}e^{-\nu x}dx \\
&= \frac{w_t}{C_t}\frac{\xi}{1+\frac{\xi}{\rho+\nu}}\int_{\tau(P)}^{+\infty}\frac{w(t-x)}{w_t}e^{-\int_{t-x}^{t}g(u)du}e^{-\nu x}dx
\end{aligned}$$

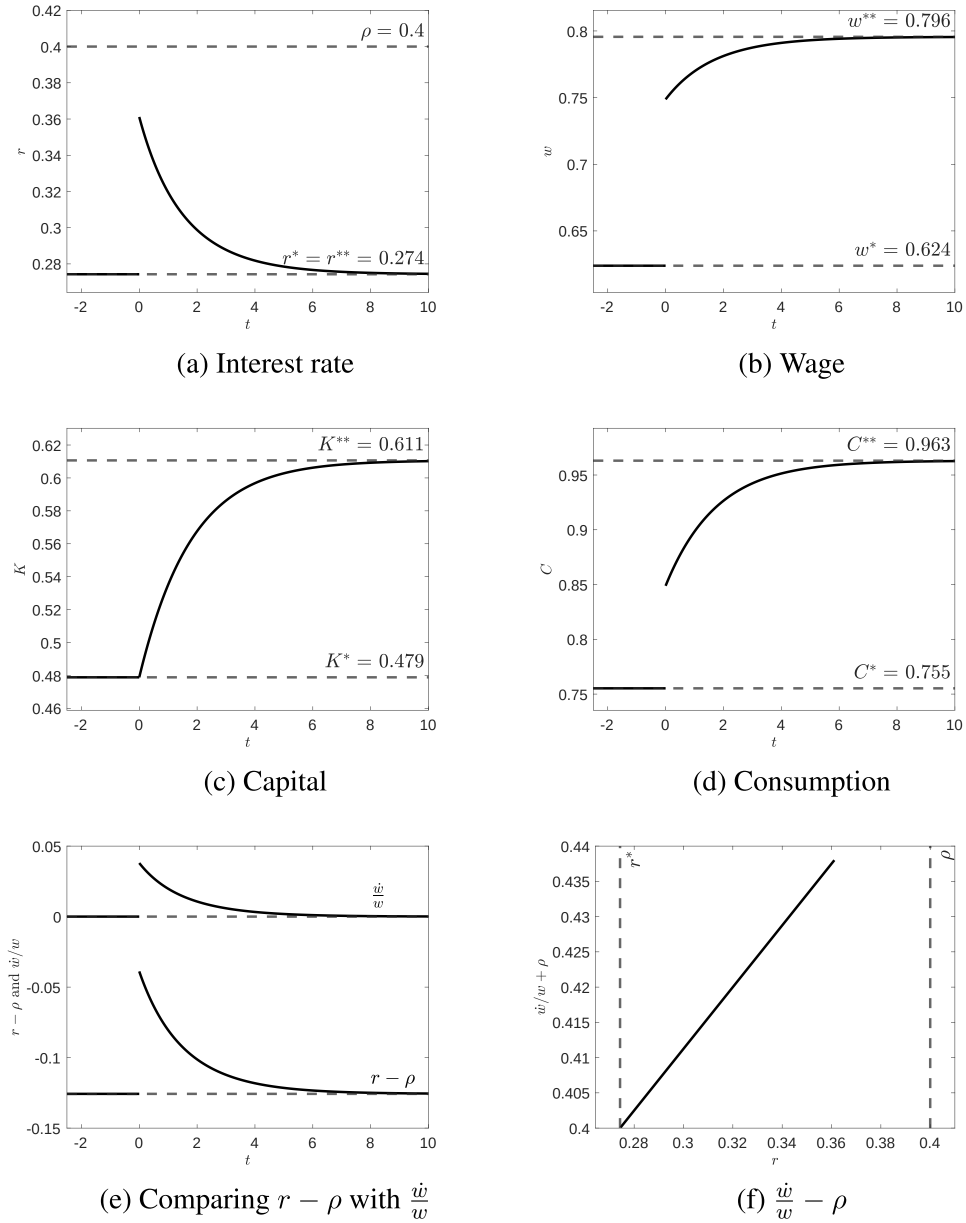
5.1.1 Numerical example

Parameters Productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$.

In this case, the productivity shock is small so that the initial interest rate is less than ρ. Hence, we can apply Lemma 2 to pin down the time path of the aggregate variables. Figure 3 plots the results. Panel (a) shows that interest rate jumps up on impact, and then drifts down to its old steady state level. Panel (b) and (d) show that wage and aggregate consumption jump up on impact, and then continues to increase to their new steady state levels. Finally, in panel (d), we see that aggregate capital increases continuously over time.

Next, we examine the evolution of consumption inequality over time. We focus on three measures – consumption gap, wage gap, and discounting gap defined in Lemma 7. Figure 4 plots these measures as a function of time. Panel (a) depicts the consumption gap as a function of time, where each line corresponds to a different τ. For a given τ, the consumption gap first jumps up, then drifts down continuously until $t = \tau$. At time τ, consumption gap shrinks discontinuously, and then gradually increases to the old steady state level. These are consistent with Lemma 7. In Figure 5, we plot the consumption inequality measures as a function of the population quantiles (ranked from low consumption to high consumption). Panel (a) plots the consumption ratio between low income household and high income household on the y-axis, where each line corresponds to a different time t. The solid line represents the steady state. We see that for a given quantile in the population, the consumption ratio drops on impact, then increases, and finally decreases to the old steady state level. The results are consistent with Figure 4. In Figure 6, we plot the CDF of the consumption ratio, where each line corresponds to a different time t.

Figure 3: Transitional dynamics of aggregate variables, with log utility and small permanent shock to A.



(a) Interest rate

(b) Wage

(c) Capital

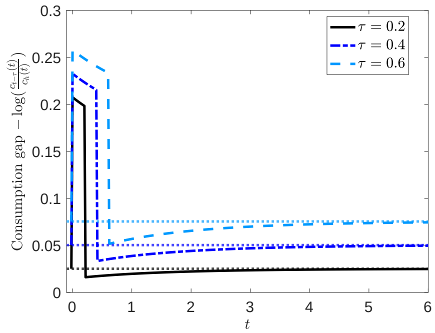
(d) Consumption

(e) Comparing $r - \rho$ with $\frac{\dot{w}}{w}$

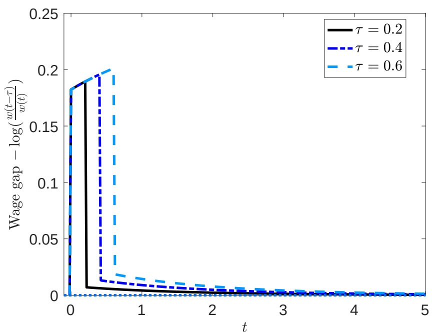
(f) $\frac{\dot{w}}{w} - \rho$

This figure plots the transitional dynamics of the aggregate variables where productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Households have log utility. And the parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$.

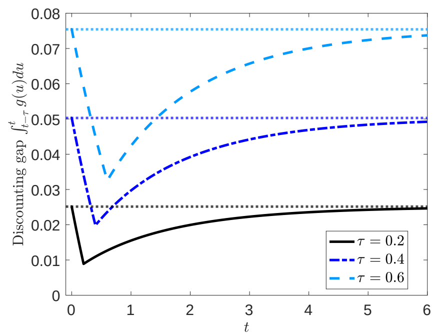
Figure 4: Consumption inequality as function of time, with log utility and small permanent shock to A.



(a) Consumption gap



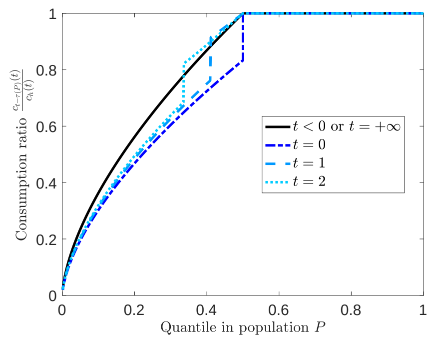
(b) Wage gap



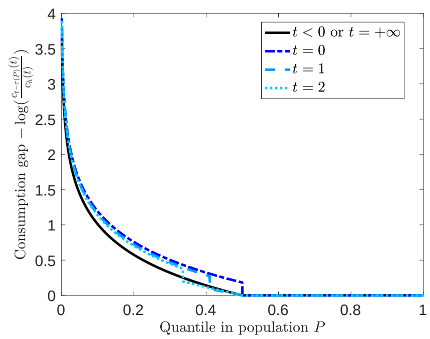
(c) Discounting gap

This figure plots the transitional dynamics of consumption inequality. Households have log utility, and the shock is a permanent increase in productivity from $A^* = 1$ to $\tilde{A} = 1.2$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$. In each panel, we plot the corresponding inequality measure on the y-axis and time t on the x-axis, and each line corresponds to a specific τ. Panel (a)-(c) plot the consumption gap, wage gap and discounting gap, respectively.

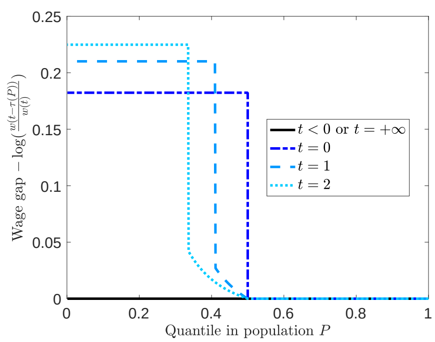
Figure 5: Consumption inequality as function of population quantiles, with log utility and small permanent shock to A.



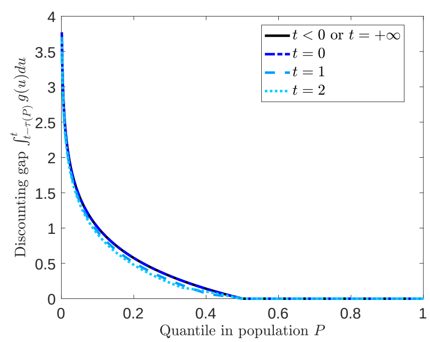
(a) Consumption ratio



(b) Consumption gap



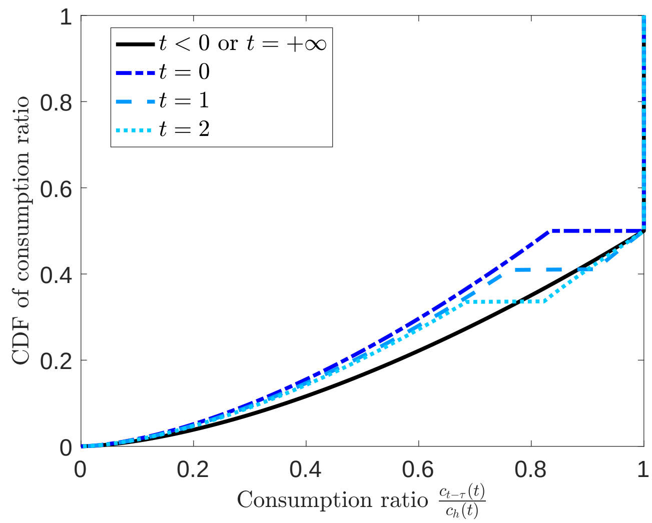
(c) Wage gap



(d) Discounting gap

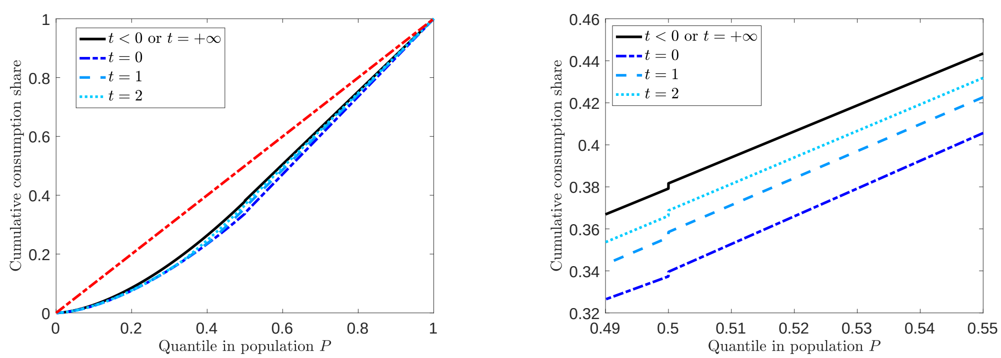
This figure plots the consumption inequality as a function of population quantiles. Households have log utility, and the shock is a permanent increase in productivity from $A^* = 1$ to $\tilde{A} = 1.2$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$. In each panel, we plot the corresponding inequality measure on the y-axis and quantiles in the population (ranked from low consumption to high consumption) on the x-axis, and each line corresponds to a specific time t. $t < 0$ or $t = +\infty$ corresponds to the steady state. Panel (a) plots the consumption ratio between a low income household with τ and a high income household. Panel (b)-(c) plot the consumption gap, wage gap and discounting gap, respectively.

Figure 6: CDF of consumption ratio between low income household and high income household, with log utility and small permanent shock to A.



This figure plots the CDF of the consumption ratio between low income household and high income household. Households have log utility, and the shock is a permanent increase in productivity from $A^* = 1$ to $\tilde{A} = 1.2$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$. Each line in the figure corresponds to a specific time t. $t < 0$ or $t = +\infty$ corresponds to the steady state.

Figure 7: Lorenz curve, with log utility and small permanent shock to A.



This figure plots the Lorenz curve. Households have log utility, and the shock is a permanent increase in productivity from $A^* = 1$ to $\tilde{A} = 1.2$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$. Each line in the figure corresponds to a specific time t. $t < 0$ or $t = +\infty$ corresponds to the steady state.

5.2 Permanent Decrease in A

Assume that, at $t = 0$, productivity decreases from A^* to $\tilde{A}$. Then the initial interest rate upon the shock, $r(0) < r^*$.

Lemma 8 (Transitional dynamics under log utility after a permanent decrease in A)**.** *Suppose the households have log utility and a permanent shock decreases productivity from A^* to $\tilde{A}$. Further impose Assumption 1 and 3. Then at any time $t \geq 0$,*

1. *Aggregate capital is*

$$K_t = \left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{\frac{1}{1-\theta}}$$

where

$$\begin{aligned} a &= \left(\frac{\xi}{\rho + \nu + \xi}(1-\theta) + \theta\right)\tilde{A} \\ b &= \delta + \rho + \nu \end{aligned}$$

Equilibrium interest rate is

$$r_t = \theta\tilde{A}\left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{-1} - \delta$$

Equilibrium wage is

$$w_t = (1-\theta)\tilde{A}\left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}}$$

2. $r_t \leq \rho$.

3. *Aggregate capital is strictly decreasing over time, thus equilibrium interest rate is strictly increasing over time, and equilibrium wage is strictly decreasing over time.*

Proof. The economy starts with capital K^* at time 0. Conjecture that $r_t \leq \rho, \forall t \geq 0$.

From equation (65), the aggregate capital supply equation can be expressed as

$$K_t = \left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{\frac{1}{1-\theta}} \tag{67}$$

where

$$
\begin{aligned}
a &= \left(\frac{\xi}{\rho+\nu+\xi}\left(1-\theta\right)+\theta\right)\tilde{A} \\
b &= \delta+\rho+\nu \\
\left(K^{*}\right)^{1-\theta} &= \frac{\theta A^{*}}{r^{*}+\delta} \\
r^{*} &= \frac{\theta\left(\nu+\rho\right)\left(\nu+\rho+\xi\right)-\xi\left(1-\theta\right)\delta}{\xi+\theta\left(\nu+\rho\right)}
\end{aligned}
$$

The equilibrium interest rate is thus

$$
r_t = \theta\tilde{A}\left(\frac{a}{b}+\left(\left(K^{*}\right)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}-\delta, \forall t\geq 0 \tag{68}
$$

and equilibrium wage satisfies

$$
w_t = \left(1-\theta\right)\tilde{A}\left(\frac{a}{b}+\left(\left(K^{*}\right)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}} \tag{69}
$$

Next, we examine the dynamics of capital, interest rate and wage. From equation (67),

$$
\dot{K}_t = \left(\frac{a}{b}+\left(\left(K^{*}\right)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}}e^{-(1-\theta)bt}b\left(\frac{a}{b}-\left(K^{*}\right)^{1-\theta}\right)<0
$$

where the last inequality follows from Lemma 9.

The dynamics of interest rate is thus

$$
\dot{r}_t = \theta\left(\theta-1\right)\tilde{A}K_t^{\theta-2}\dot{K}_t>0
$$

So interest rate is monotonically increasing over time. And the dynamics of wage is

$$
\dot{w}_t = \left(1-\theta\right)\tilde{A}\theta K_t^{\theta-1}\dot{K}_t<0
$$

Hence, aggregate capital is strictly decreasing over time, interest rate is strictly increasing, and wage is strictly decreasing.

Now we check the terminal conditions. As $t \to +\infty$, we have

$$\begin{aligned}
\lim_{t \to +\infty} K_t &= \left(\frac{a}{b}\right)^{\frac{1}{1-\theta}} = K^{**} \\
\lim_{t \to +\infty} r_t &= r^{**} \\
\lim_{t \to +\infty} w_t &= w^{**}
\end{aligned}$$

Hence, the aggregate capital, interest rate, and wage paths derived above are consistent with the terminal conditions.

Finally, since the initial interest rate upon the shock is $r(0) < r^* < \rho$ and in the new steady state $r(+\infty) = r^{**} = r^* < \rho$, we have $r_t \leq \rho, \forall t \geq 0$, which verifies the conjecture. $\square$

Lemma 9. *Suppose the households have log utility, and a permanent shock decreases productivity from A^* to $\tilde{A}$. Further impose Assumption 1. Then*

$$\frac{a}{b} < (K^*)^{1-\theta}$$

where a and b are defined in Lemma 8.

Proof. Since $\tilde{A} < A^*$, and by the definition of a and b in Lemma 8,

$$\begin{aligned}
\frac{a}{b} &< \frac{\frac{\xi}{\rho+\nu+\xi}(1-\theta)+\theta}{\delta+\rho+\nu} A^* = \frac{(\nu+\rho-r^*)\frac{\xi}{(\nu+\rho-r^*)(\rho+\nu+\xi)}(1-\theta)+\theta}{\delta+\rho+\nu} A^* \\
&= \frac{(\nu+\rho-r^*)\frac{\theta}{r^*+\delta}+\theta}{\delta+\rho+\nu} A^* = \frac{\theta A^*}{r^*+\delta} \\
&= (K^*)^{1-\theta}
\end{aligned}$$

Note that the second line follows from Assumption 1. In particular, Assumption 1 guarantees a unique equilibrium interest rate r^* that supports the original partial insurance equilibrium, and more importantly, this interest rate satisfies the condition

$$\frac{\xi}{(\nu+\rho-r^*)(\rho+\nu+\xi)} = \frac{\theta}{(1-\theta)(r^*+\delta)}$$

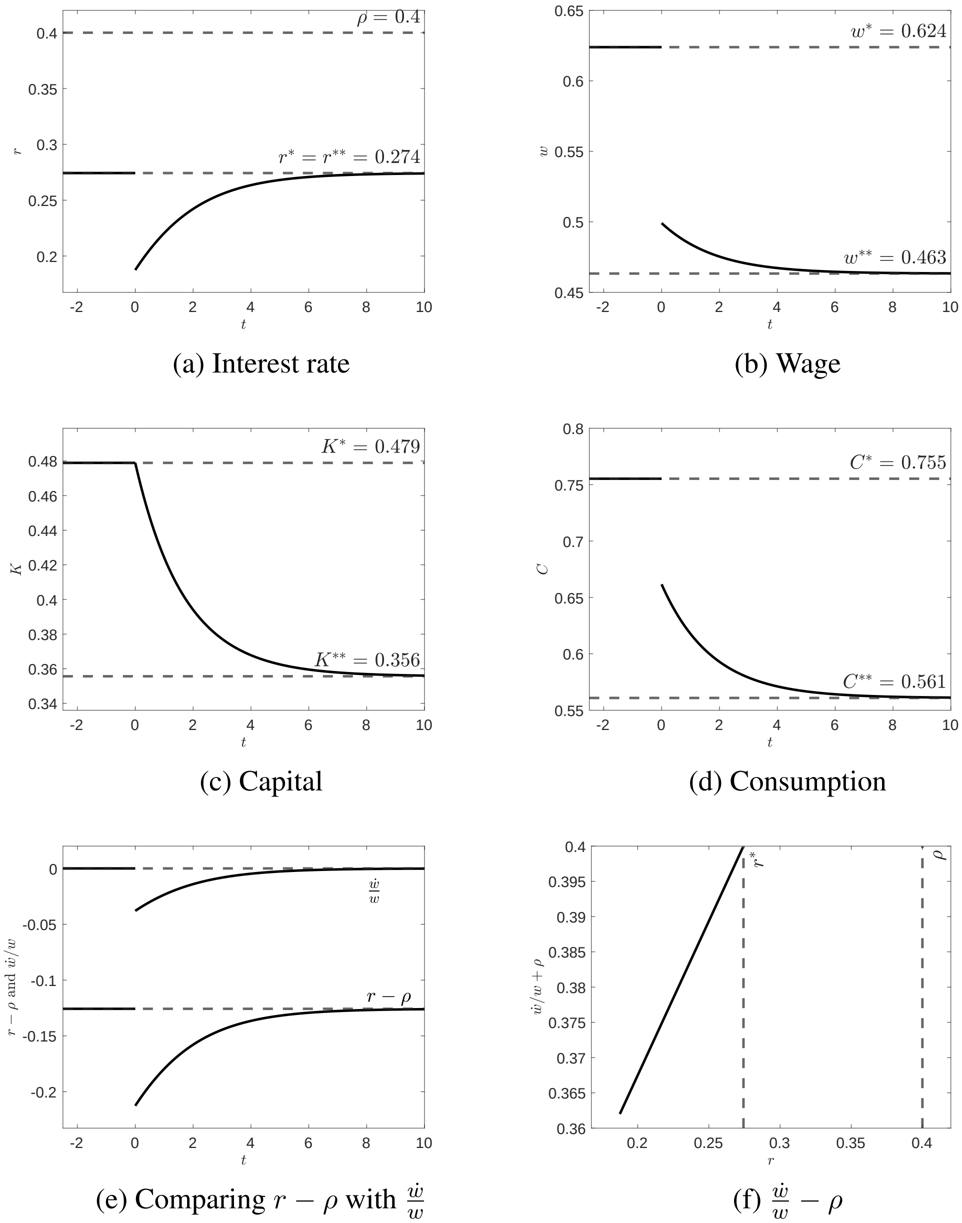
$\square$

The analysis of consumption inequality and the consumption gap in the case of a decline in aggregate productivity is completely symmetric to that in the previous subsection. We now display a numerical example for such a productivity decline.

5.2.1 Numerical example

Parameters Productivity permanently decreases from $A^* = 1$ to $\tilde{A} = 0.8$. Other parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$.

Figure 8: Transitional dynamics of aggregate variables, with log utility and a permanent decrease in A.



(a) Interest rate

(b) Wage

(c) Capital

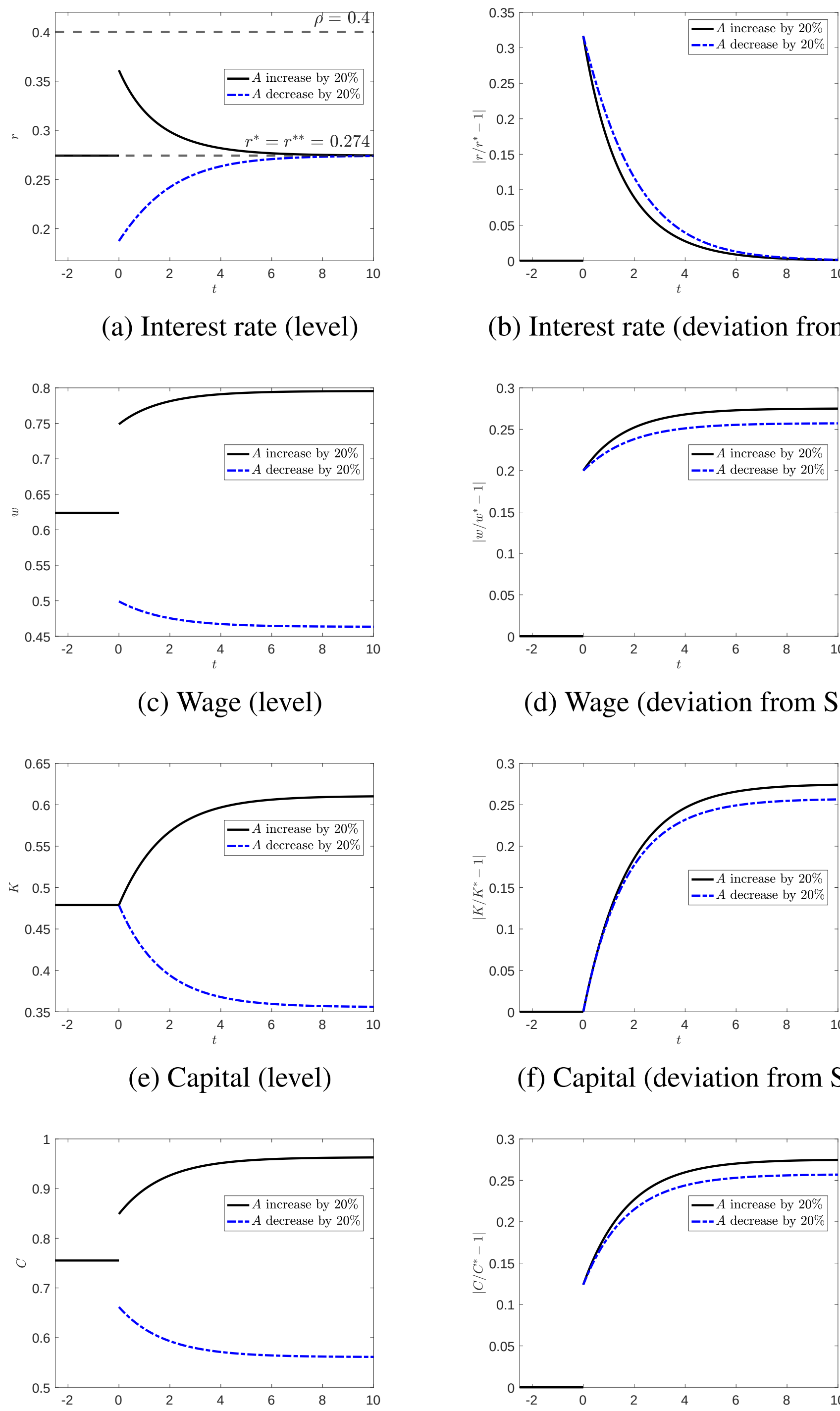
(d) Consumption

(e) Comparing $r - \rho$ with $\frac{\dot{w}}{w}$

(f) $\frac{\dot{w}}{w} - \rho$

This figure plots the transitional dynamics of the aggregate variables where productivity permanently decreases from $A^* = 1$ to $\tilde{A} = 0.8$. Households have log utility. And the parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$.

Figure 9: Comparing transitional dynamics of aggregate variables under different permanent shocks to A.



(a) Interest rate (level) (b) Interest rate (deviation from SS)

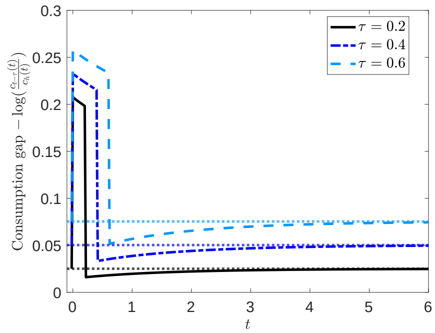
(c) Wage (level) (d) Wage (deviation from SS)

(e) Capital (level) (f) Capital (deviation from SS)

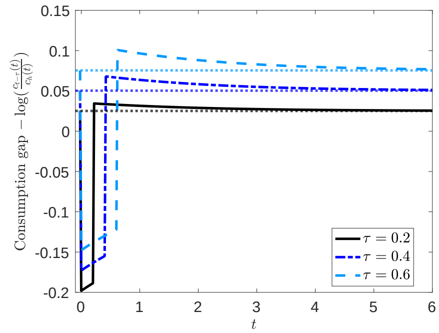
(g) Consumption (level) (h) Consumption (deviation from SS)

This figure compares the transitional dynamics of the aggregate variables under two cases: (1) productivity permanently increases by 20%, and (2) productivity permanently decreases by 20%. Households have log utility. And the parameter values are $\theta = 0.25, \delta = 0.16, \nu = 0.2, \xi = 0.2, \rho = 0.4$.

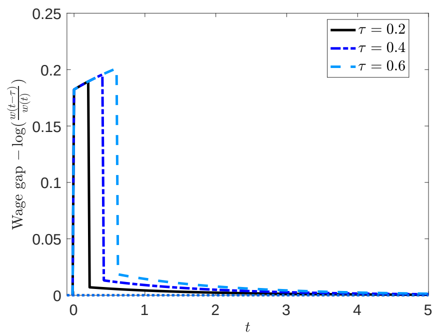
Figure 10: Consumption inequality as function of time.



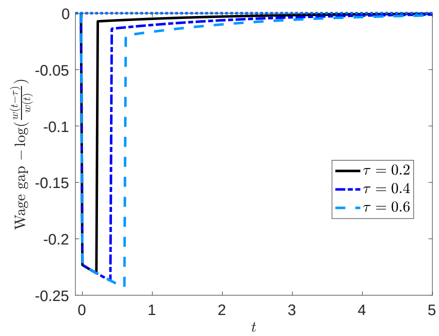
(a) Consumption gap, A increases by 20%



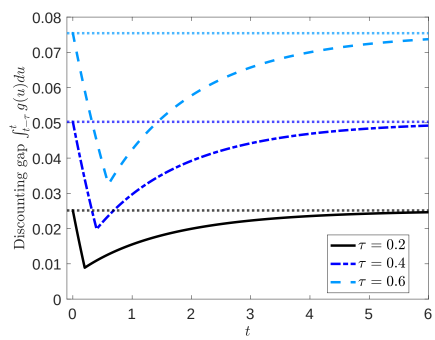
(b) Consumption gap, A decreases by 20%



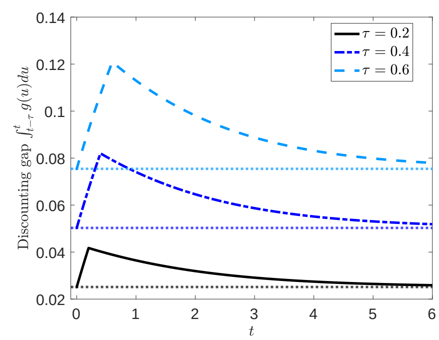
(c) Wage gap, A increases by 20%



(d) Wage gap, A decreases by 20%



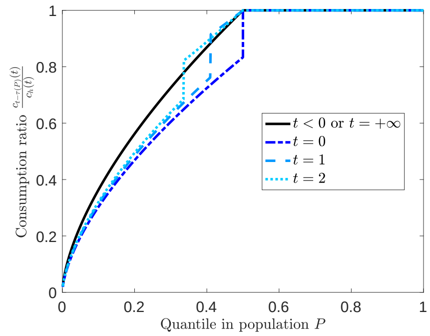
(e) Discounting gap, A increases by 20%



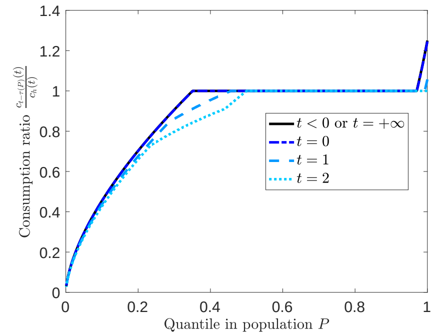
(f) Discounting gap, A decreases by 20%

This figure plots the transitional dynamics of consumption inequality under two cases: (1) productivity permanently increases by 20%, and (2) productivity permanently decreases by 20%.

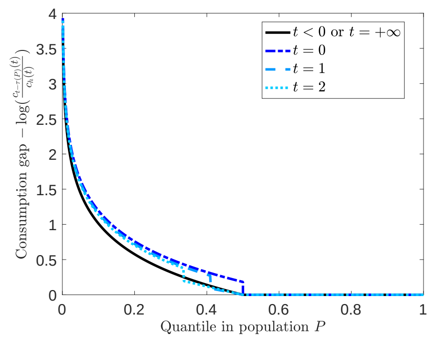
Figure 11: Consumption inequality as function of population quantiles.



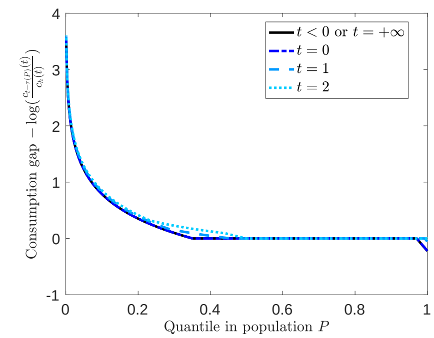
(a) Consumption ratio, A increases by 20%



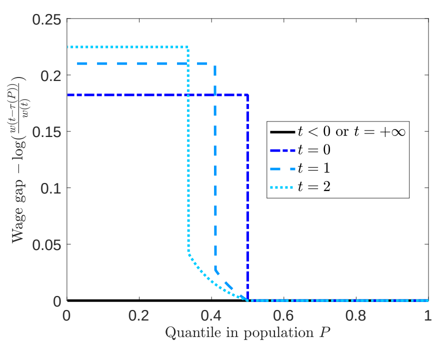
(b) Consumption ratio, A decreases by 20%



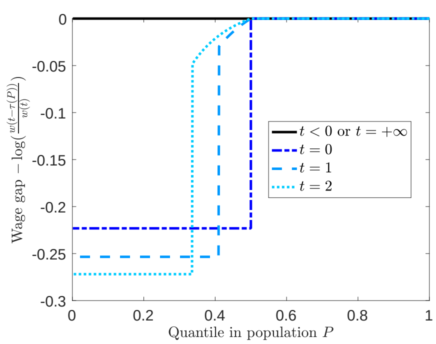
(c) Consumption gap, A increases by 20%



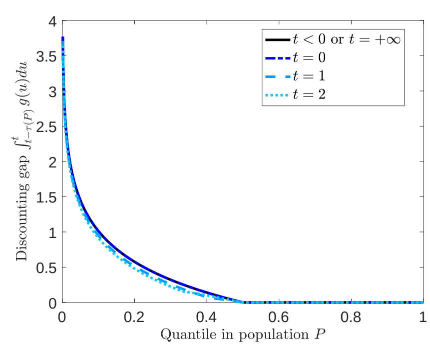
(d) Consumption gap, A decreases by 20%



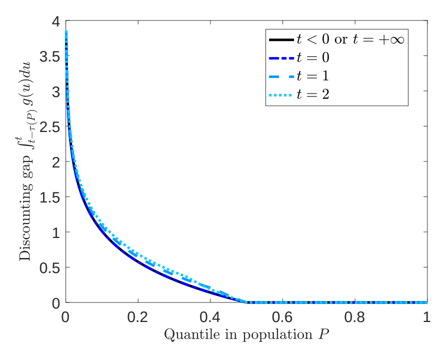
(e) Wage gap, A increases by 20%



(f) Wage gap, A decreases by 20%



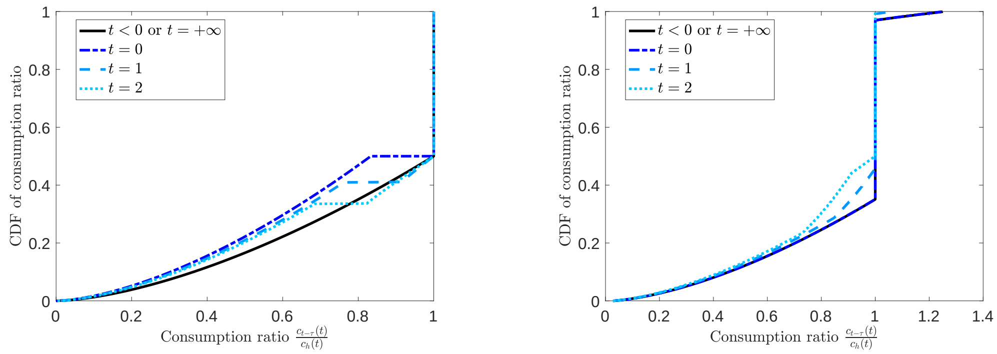
(g) Discounting gap, A increases by 20%



(h) Discounting gap, A decreases by 20%

This figure plots the consumption inequality as a function of population quantiles, under two cases: (1) productivity permanently increases by 20%, and (2) productivity permanently decreases by 20%

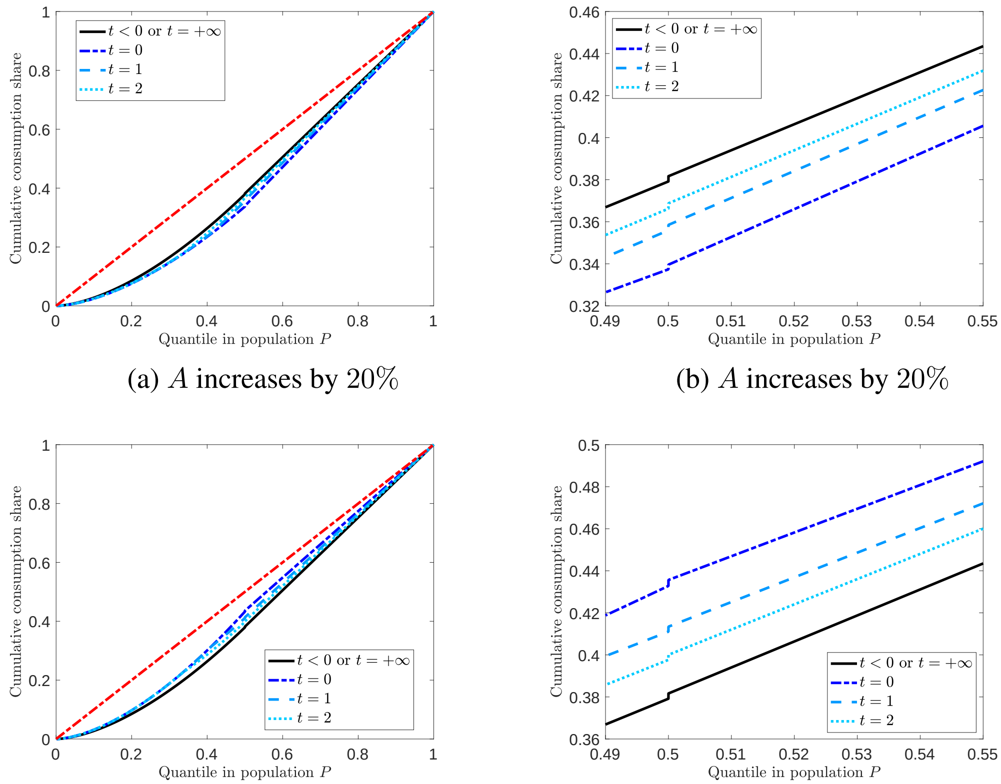
Figure 12: CDF of consumption ratio between low income household and high income household.



(a) A increases by 20% (b) A decreases by 20%

This figure plots the CDF of the consumption ratio between low income household and high income household, under two cases: (1) productivity permanently increases by 20%, and (2) productivity permanently decreases by 20%.

Figure 13: Lorenz curve.



(a) A increases by 20%

(b) A increases by 20%

(c) A decreases by 20%

(d) A decreases by 20%

This figure plots the Lorenz curve under two cases: (1) productivity permanently increases by 20%, and (2) productivity permanently decreases by 20%.

6 Conclusion

In this paper we have analytically characterized the transition dynamics in a neoclassical production economy with idiosyncratic income shocks and long-term one-sided limited commitment contracts. For an important special case (log-utility, two income state, zero income in the lower state) the transition path induced by an unexpected productivity shock equilibrium is unique and can be given in closed form, both for the macroeconomic variables as well as the nondegenerate consumption distribution, which displays partial consumption insurance against the idiosyncratic income shocks.

Given these findings, we would identify two immediately relevant next questions. First, on account of our use of continuous time, the endogenous optimal contract length is analytically tractable even outside the special case we have focused on thus far, and it will be important to generalize our findings to the more general case.

Second, thus far we have focused on an environment that has idiosyncratic, but no aggregate shocks, rendering the macroeconomic dynamics deterministic. Given our sharp analytical characterization of the equilibrium in the absence of aggregate shocks, we conjecture that the economy with aggregate shocks might be at least partially analytically tractable as well. We view these questions as important topics for future research.

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A Proofs

A.1 Proof of Lemma 1

Proof. Generally, let $\mu \geq 0$ denote the Lagrange multiplier on the budget constraint (5), λ the Lagrange multiplier on the borrowing constraint (6) and $\vartheta \geq 0$ the Lagrange multiplier on the constraint $\tilde{k} \geq 0$. Then the Lagrangian for the optimal contracting problem of definition 1 is

$$\begin{aligned} \mathcal{L} &= u(c) + \dot{U}_t(k;z) + U_t'(k,z)\,x + p\left(U_t\left(\tilde{k},\tilde{z}\right) - U_t(k;z)\right) \\ &\quad -\mu\left(c + x + p\left(\tilde{k} - k\right) - r_t k - w_t z\right) \\ &\quad +\lambda x + \vartheta \tilde{k} \end{aligned} \tag{A.1}$$

The FOC's are

$$\frac{\partial}{\partial c}: \quad u'(c) = \mu \tag{A.2}$$

$$\frac{\partial}{\partial x}: \quad U_t'(k;z) = \mu - \lambda \tag{A.3}$$

$$\frac{\partial}{\partial \tilde{k}}: \quad U_t'\left(\tilde{k},\tilde{z}\right) = \mu - \frac{\vartheta}{p} \tag{A.4}$$

Note that (6) is not a constraint, when $k > 0$: in that case, $\lambda = 0$. Then, equations (A.2) and (A.3) imply that

$$u'(c) = U_t'(k;z) \tag{A.5}$$

When productivity stays unchanged for an interval of time, differentiate both sides of equation (A.5) w.r.t. time t and recognize that $\dot{k}_t = x$ to obtain

$$u''(c)\,\dot{c} = \dot{U}_t'(k;z) + U_t''(k;z)\,x \tag{A.6}$$

where

$$
\begin{aligned}
\dot{c}_t(k;z) &\equiv \frac{\partial c_t(k;z)}{\partial t} + \frac{\partial c_t(k;z)}{\partial k} x_t(k;z) \\
U_t''(k;z) &\equiv \frac{\partial^2 U_t(k;z)}{\partial k^2} \\
\dot{U}_t'(k;z) &\equiv \frac{\partial^2 U_t(k;z)}{\partial k \partial t}
\end{aligned}
$$

The envelope condition arising from differentiating the objective (4) with respect to the state k, is

$$
\rho U_t'(k;z) = U_t''(k;z)\, x + \dot{U}_t'(k;z) - pU_t'(k;z) + \mu(p + r_t) \tag{A.7}
$$

This successively implies

$$
\begin{aligned}
\rho U_t'(k;z) &= U_t''(k;z)\, x + \dot{U}_t'(k;z) + u'(c)\, r_t \\
\rho u'(c) &= u''(c)\, \dot{c} + u'(c)\, r_t \\
\frac{\dot{c}}{c} &= \frac{u'(c)}{cu''(c)}(\rho - r_t) = -g_t
\end{aligned}
$$

and thus (32), where the second line uses the FOCs in equations (A.2) and (A.3), and the third line uses equations (A.5) and (A.6).

When $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, then[4] (32) and the budget constraint can be rewritten as the linear system of differential equations

$$
\dot{c}_t = -g_t c_t \tag{A.8}
$$

$$
\dot{k}_t = (r_t + \nu) k_t - c_t \tag{A.9}
$$

in the unknown functions c_t and k_t with the boundary condition[5] $\lim_{t\to\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t. Conjecture that the solution satisfies (33), rewritten as

$$
k_t = D_t c_t \tag{A.10}
$$

[4]Note that (32) implies that $c_s = e^{-\int_t^s g_u du} c_t$. A less formal, but more meaningful argument in terms of economic theory is thus to recognize that the budget constraint and utility maximization implies that the current capital k is equal to the net present value of all future consumption, as long as the productivity state stays unchanged. This yields $k = \int_t^{+\infty} e^{-\int_t^s (r_u+\nu) du} c_s ds = D_t c_t$.

[5]The boundary condition ensures that no capital gets wasted, and follows from utility maximization.

Differentiate (A.10) with respect to time. Note that

$$\dot{D}_t = -1 + (r_t + \nu + g_t)D_t \tag{A.11}$$

and use (A.8) to derive (A.9). The solution is valid, as long as the implied path for k_s for $s \geq t$ does not cross the upper bound $\bar{k}$. This will be true for all $k_t \in (0, \bar{\bar{k}})$ and some suitable $\bar{\bar{k}}$. □

A.2 Proof of Lemma 2

Proof. The Lagrangian, first-order conditions and envelope condition are as in the proof for Lemma 1, see (A.1), (A.2), (A.3), (A.4) and (A.7), applied to $k = 0$, $z = \zeta$ and $p = \xi$. Consider first the choice of $\tilde{k}$. The solution in Lemma 1 implies that

$$U_t'(\tilde{k}, 0) = u'\left(\frac{\tilde{k}}{D_t}\right) \tag{A.12}$$

which increases to infinity, as $\tilde{k} \to 0$. Equation (A.4) therefore implies that $\tilde{k} > 0$ and thus $\vartheta = 0$. With the first order condition (A.2), we obtain consumption smoothing

$$u'\left(\frac{\tilde{k}}{D_t}\right) = u'(c) \tag{A.13}$$

or

$$\tilde{k} = D_t c \tag{A.14}$$

Therefore, (36) follows from the budget constraint (5), provided that $x = 0$. We thus need to show that $x > 0$ is not optimal. If $x > 0$ were optimal, then (6) would not be binding, $\lambda = 0$ and consumption growth would satisfy (32). This leads to a contradiction. **WE STILL NEED TO FILL IN THE ARGUMENT, WHY WE ARRIVE AT A CONTRADICTION. A VERBAL ARGUMENT IS GIVEN IN THE TEXT, BUT WE NEED A FORMAL ARGUMENT** □

A.3 Proof of Lemma 4

Proof. Differentiate both sides of equation (56) w.r.t. t

$$\begin{aligned}
\dot{K}_t &= k_{t,t} + \int_{-\infty}^{t} \left(\dot{k}_{s,t}\psi_l(t-s) + k_{s,t}\psi_l'(t-s) \right) ds \\
&= k_{t,t}\psi_l(0) + \int_{-\infty}^{t} \left(\left(r_t + \nu - \frac{1}{D_t} \right) k_{s,t}\psi_l(t-s) - \nu k_{s,t}\psi_l(t-s) \right) ds \\
&= \frac{w_t\zeta}{1+\xi D_t} D_t \frac{\xi\nu}{\xi+\nu} + \left(r_t - \frac{1}{D_t} \right) K_t \\
&= \frac{\xi w_t}{1+\xi D_t} D_t + \left(r_t - \frac{1}{D_t} \right) K_t \\
&= \frac{\xi\left(1-\theta\right) A_t K_t^{\theta}}{1+\xi D_t} D_t + \left(\theta A_t K_t^{\theta-1} - \delta - \frac{1}{D_t} \right) K_t \\
&= \left(\frac{\xi D_t}{1+\xi D_t} \left(1-\theta\right) + \theta \right) A_t K_t^{\theta} - \left(\delta + \frac{1}{D_t} \right) K_t
\end{aligned}$$

where we have used the normalization (3) for the forth equality. □

A.4 Proof of Lemma 5

Proof. Equation (54) is a Bernoulli differential equation. Given an initial condition K_0, it can be solved in the following way:

1. Rewrite equation (54) as a linear differential equation.

   Define $X_t = K_t^{1-\theta}$, $a_t = \left(\frac{\xi D_t}{1+\xi D_t} \left(1-\theta\right) + \theta \right) A_t$, $b_t = \delta + \frac{1}{D_t}$. Then we can rewrite equation (54) as

$$\dot{X}_t + \left(1-\theta\right) b_t X_t = \left(1-\theta\right) a_t \tag{A.15}$$

2. Solve the linear differential equation (A.15).

   Multiply both sides of equation (A.15) by $e^{(1-\theta)\int_0^t b(u)du}$ to get

$$\frac{d\left(e^{(1-\theta)\int_0^t b(u)du} X_t \right)}{dt} = \left(1-\theta\right) e^{(1-\theta)\int_0^t b(u)du} a_t$$

Integrate both sides from 0 to t.

$$e^{(1-\theta)\int_0^t b(u)du} X_t = X(0) + \int_0^t (1-\theta) e^{(1-\theta)\int_0^s b(u)du} a(s)\,ds$$

Hence, given the boundary condition $X(0)$,

$$X_t = e^{-(1-\theta)\int_0^t b(u)du} X(0) + (1-\theta)\int_0^t e^{(1-\theta)\int_t^s b(u)du} a(s)\,ds \quad \text{(A.16)}$$

3. Substitute the definition of X_t into equation (A.16)

$$\begin{aligned} K_t &= \left(e^{-(1-\theta)\int_0^t b(u)du} K_0^{1-\theta} + (1-\theta)\int_0^t e^{(1-\theta)\int_t^s b(u)du} a(s)\,ds\right)^{\frac{1}{1-\theta}} \\ &= \left(e^{-(1-\theta)\int_0^t b(u)du} K_0^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)\int_s^t b(u)du} a(s)\,ds\right)^{\frac{1}{1-\theta}} \end{aligned}$$

□

A.5 Proof of Proposition 2

Proof. The economy starts with capital K^* at time 0. Conjecture that $r_t \leq \rho, \forall t \geq 0$.

From equation (65), the aggregate capital supply equation can be expressed as

$$K_t = \left(\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right) e^{-(1-\theta)bt}\right)^{\frac{1}{1-\theta}} \quad \text{(A.17)}$$

where

$$\begin{aligned} a &= \left(\frac{\xi}{\rho+\nu+\xi}(1-\theta)+\theta\right)\tilde{A} \\ b &= \delta+\rho+\nu \\ (K^*)^{1-\theta} &= \frac{\theta A^*}{r^*+\delta} \\ r^* &= \frac{\theta(\nu+\rho)(\nu+\rho+\xi)-\xi(1-\theta)\delta}{\xi+\theta(\nu+\rho)} \end{aligned}$$

The equilibrium interest rate is thus

$$r_t = \theta\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}-\delta, \forall t\geq 0 \tag{A.18}$$

and equilibrium wage satisfies

$$w_t = (1-\theta)\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}} \tag{A.19}$$

Next, we examine the dynamics of capital, interest rate and wage. From equation (A.17),

$$\dot{K}_t = \left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{\frac{\theta}{1-\theta}}e^{-(1-\theta)bt}b\left(\frac{a}{b}-(K^*)^{1-\theta}\right)>0$$

where the last inequality follows from Lemma 6.

The dynamics of interest rate is thus

$$\dot{r}_t = \theta(\theta-1)\tilde{A}K_t^{\theta-2}\dot{K}_t<0$$

So $r_t\leq\rho,\forall t\geq 0$ (which verifies the conjecture), and interest rate is monotonically decreasing over time.

And the dynamics of wage is

$$\dot{w}_t = (1-\theta)\tilde{A}\theta K_t^{\theta-1}\dot{K}_t>0$$

Hence, aggregate capital is strictly increasing over time, interest rate is strictly decreasing, and wage is strictly increasing.

Finally, we check the terminal conditions. As $t\to+\infty$, we have

$$\begin{aligned}\lim_{t\to+\infty}K_t &= \left(\frac{a}{b}\right)^{\frac{1}{1-\theta}}=K^{**}\\ \lim_{t\to+\infty}r_t &= r^{**}\\ \lim_{t\to+\infty}w_t &= w^{**}\end{aligned}$$

Hence, the aggregate capital, interest rate, and wage paths derived above are consistent with the terminal conditions. □

A.6 Proof of Lemma 6

Proof. Since $\tilde{A} > A^*$, and by the definition of a and b in Lemma 2,

$$\begin{aligned}
\frac{a}{b} &> \frac{\frac{\xi}{\rho+\nu+\xi}(1-\theta)+\theta}{\delta+\rho+\nu}A^* = \frac{(\nu+\rho-r^*)\frac{\xi}{(\nu+\rho-r^*)(\rho+\nu+\xi)}(1-\theta)+\theta}{\delta+\rho+\nu}A^* \\
&= \frac{(\nu+\rho-r^*)\frac{\theta}{r^*+\delta}+\theta}{\delta+\rho+\nu}A^* = \frac{\theta A^*}{r^*+\delta} \\
&= (K^*)^{1-\theta}
\end{aligned}$$

Note that the second line follows from Assumption 1. In particular, Assumption 1 guarantees a unique equilibrium interest rate r^* that supports the original partial insurance equilibrium, and more importantly, this interest rate satisfies the condition

$$\frac{\xi}{(\nu+\rho-r^*)(\rho+\nu+\xi)} = \frac{\theta}{(1-\theta)(r^*+\delta)}$$

□

A.7 Proof of Lemma 7

Proof. According to equations (62) and (63) characterizing the optimal consumption allocation under log utility, the consumption of type 1 and type 2 households are

$$\begin{aligned}
c_{t;h} &= \frac{w_t z}{1+\frac{\xi}{\rho+\nu}} \\
c_{t-\tau,t} &= c_h(t-\tau)\,e^{-\int_{t-\tau}^{t} g(u)du}
\end{aligned}$$

Then the consumption ratio between low income household and high income household can be expressed as

$$\frac{c_{t-\tau,t}}{c_{t;h}} = \frac{w(t-\tau)}{w_t}e^{-\int_{t-\tau}^{t} g(u)du} < 1$$

Hence, this consumption ratio is always less than 1, which implies that any low income household always consumes less than the high income household.

The consumption gap is thus

$$-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) \quad = \quad -\log\left(\frac{w(t-\tau)}{w_t}\right)+\int_{t-\tau}^{t} g(u)\,du$$

Also note that, according to Lemma 2, $w^* < w_t < w(t+dt), \forall t \geq 0$, which implies $c_h^* < c_{t;h} < c_h(t+dt), \forall t \geq 0$. And $r^* < r(0), r_t > r(t+dt) \geq r^*, t \geq 0$.

Given a $\tau \geq 0$, we want to examine the change in the consumption gap from time t to time $t+dt$ for some small $dt > 0$ and some $t \geq 0$. Consider the following three cases.

- $t = -dt < 0$. Then

$$\begin{aligned}
d\left(-\log\left(\frac{w(t-\tau)}{w_t}\right)\right) &= -\log\left(\frac{w^*}{w(0)}\right)+\log\left(\frac{w^*}{w^*}\right) = -\log\left(\frac{w^*}{w(0)}\right) > 0 \\
d\left(\int_{t-\tau}^{t} g(u)\,du\right) &= (\rho - r^*)(\tau - dt) + (\rho - r(0))\,dt - (\rho - r^*)\,\tau \\
&= (r^* - r(0))\,dt < 0
\end{aligned}$$

which implies

$$d\left(-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right)\right) \quad > \quad 0$$

Hence, from time $-dt$ (right before the shock) to time 0 (immediately after the shock), the wage gap widens discontinuously while the discounting gap shrinks continuously. This implies that the consumption gap widens discontinuously.

- $t \in [0, \tau - dt)$. In this case, $t - \tau < 0, t + dt - \tau < 0$. Then

$$\begin{aligned}
\frac{d\left(-\log\left(\frac{w(t-\tau)}{w_t}\right)\right)}{dt} &= \frac{-\log\left(\frac{w^*}{w(t+dt)}\right)+\log\left(\frac{w^*}{w_t}\right)}{dt} = \frac{\log\left(\frac{w(t+dt)}{w_t}\right)}{dt} = \frac{d(\log(w_t))}{dt} \\
&= \theta b\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}\left(\frac{a}{b}-(K^*)^{1-\theta}\right)e^{-(1-\theta)bt} > 0 \\
\frac{d\left(\int_{t-\tau}^{t} g(u)\,du\right)}{dt} &= g_t - g(t-\tau) = r^* - r_t \\
&= \theta\tilde{A}\left(\frac{a}{b}\right)^{-1} - \theta\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1} \leq 0
\end{aligned}$$

which implies

$$\frac{d\left(-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right)\right)}{dt} = \frac{\theta b\left(\frac{a}{b}-(K^*)^{1-\theta}\right)e^{-(1-\theta)bt}}{\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}}\left(1-\frac{\tilde{A}}{a}\right)<0$$

Hence, for any $t<\tau-dt$, the wage gap widens continuously while the discounting gap shrinks continuously. The latter effect dominates, so consumption gap shrinks continuously.

- $t=\tau-dt$. In this case, $t-\tau<0, t+dt-\tau=0$. Then

$$\begin{aligned}
& d\left(-\log\left(\frac{w(t-\tau)}{w_t}\right)\right) \\
= & -\log\left(\frac{w(0)}{w(t+dt)}\right)+\log\left(\frac{w^*}{w_t}\right)=d(\log(w_t))+\log\left(\frac{w^*}{w(0)}\right) \\
= & \left[\theta b\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}\left(\frac{a}{b}-(K^*)^{1-\theta}\right)e^{-(1-\theta)bt}\right]dt+\log\left(\frac{w^*}{w(0)}\right) \\
< & \ 0 \\
& \frac{d\left(\int_{t-\tau}^{t} g(u)\,du\right)}{dt} \\
= & \ g_t-g(t-\tau)=r^*-r_t \\
= & \ \theta\tilde{A}\left(\frac{a}{b}\right)^{-1}-\theta\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}\le 0
\end{aligned}$$

which implies

$$d\left(-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right)\right) < 0$$

Hence, from $t=\tau-dt$ to $t=\tau$, the wage gap shrinks discontinuously and the discounting gap shrinks continuously, which implies that the consumption gap shrinks discontinuously.

- $t \geq \tau$. In this case, $t-\tau \geq 0, t+dt-\tau > 0$. Then

$$\begin{aligned}
\frac{d\left(-\log\left(\frac{w(t-\tau)}{w_t}\right)\right)}{dt} &= \frac{-\log\left(\frac{w(t+dt-\tau)}{w(t+dt)}\right)+\log\left(\frac{w(t-\tau)}{w_t}\right)}{dt} \\
&= \theta b\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}\left(\frac{a}{b}-(K^*)^{1-\theta}\right)e^{-(1-\theta)bt} \\
&\quad -\theta b\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}\right)^{-1}\left(\frac{a}{b}-(K^*)^{1-\theta}\right)e^{-(1-\theta)b(t-\tau)} \\
&< 0 \\
\frac{d\left(\int_{t-\tau}^{t} g(u)\,du\right)}{dt} &= g_t - g(t-\tau) = r(t-\tau) - r_t \\
&= \theta\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}\right)^{-1} \\
&\quad -\theta\tilde{A}\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1} \\
&> 0
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d\left(-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right)\right)}{dt} &= \left(-\theta\left(\tilde{A}-a\right)\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1}-\theta b\right) \\
&\quad -\left(-\theta\left(\tilde{A}-a\right)\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}\right)^{-1}-\theta b\right) \\
&= -\theta\left(\tilde{A}-a\right)\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)bt}\right)^{-1} \\
&\quad +\theta\left(\tilde{A}-a\right)\left(\frac{a}{b}+\left((K^*)^{1-\theta}-\frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}\right)^{-1} \\
&> 0
\end{aligned}$$

Hence, for $t > \tau$, the wage shrinks continuously while the discounting gap widens continuously. The latter effect dominates, so consumption gap widens continuously.

Finally, in the limiting case $t \to +\infty$, the consumption gap is

$$\begin{aligned}
\lim_{t\to+\infty} -\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) &= \lim_{t\to+\infty} -\log\left(\frac{w\left(t-\tau\right)}{w_t}\right) + \lim_{t\to+\infty}\int_{t-\tau}^{t} g\left(u\right)du \\
&= \lim_{t\to+\infty}\int_{t-\tau}^{t} g\left(u\right)du \\
&= \left(\rho - r^*\right)\tau \\
&= -\log\left(\frac{c^*_{-\tau}}{c^*_h}\right)
\end{aligned}$$

Hence, the consumption gap at the new steady state is the same as that at the old steady state. □

B Other Detailed Derivations

B.1 Closed form solution of the consumption gap

- $t = -dt$.

$$\int_{t-\tau}^{t} g\left(u\right)du = \left(\rho - r^*\right)\tau, -\log\left(\frac{w\left(t-\tau\right)}{w_t}\right) = 0$$

Consumption gap is thus

$$-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) = \left(\rho - r^*\right)\tau$$

- $t \in [0, \tau]$.

$$\begin{aligned}
\int_{t-\tau}^{t} g(u)\,du &= \rho\tau - \int_{t-\tau}^{t} r(u)\,du = \rho\tau - \int_{t-\tau}^{0} r^{*}du - \int_{0}^{t} r(u)\,du \\
&= \rho\tau - (\tau - t)\,r^{*} - \int_{0}^{t} d\left[\frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{a}{b}e^{(1-\theta)bu} + (K^{*})^{1-\theta} - \frac{a}{b}\right) - \delta u\right] \\
&= (\rho - r^{*})\,\tau + (r^{*}+\delta)\,t - \frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{\frac{a}{b}e^{(1-\theta)bt} + (K^{*})^{1-\theta} - \frac{a}{b}}{(K^{*})^{1-\theta}}\right) \\
&= (\rho - r^{*})\,\tau + (r^{*}+\delta)\,t \\
&\quad - \frac{\theta\tilde{A}}{(1-\theta)\,a}\left((1-\theta)\,bt + \log\left(\frac{\frac{a}{b} + \left((K^{*})^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{(K^{*})^{1-\theta}}\right)\right) \\
&= (\rho - r^{*})\,\tau + \left(r^{*} + \delta - \theta\tilde{A}\frac{b}{a}\right)t \\
&\quad - \frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{\frac{a}{b} + \left((K^{*})^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{(K^{*})^{1-\theta}}\right) \\
-\log\left(\frac{w(t-\tau)}{w_t}\right) &= -\log\left(\frac{w^{*}}{w_t}\right) \\
&= \log\left(\frac{\tilde{A}}{A^{*}}\right) - \theta\log K^{*} + \frac{\theta}{1-\theta}\log\left(\frac{a}{b} + \left((K^{*})^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}\right) \\
&= \log\left(\frac{\tilde{A}}{A^{*}}\right) + \frac{\theta}{1-\theta}\log\left(\frac{\frac{a}{b} + \left((K^{*})^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{(K^{*})^{1-\theta}}\right)
\end{aligned}$$

Consumption gap is thus

$$\begin{aligned}
-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) &= -\log\left(\frac{w(t-\tau)}{w_t}\right) + \int_{t-\tau}^{t} g(u)\,du \\
&= (\rho - r^{*})\,\tau + \left(r^{*} + \delta - \theta\tilde{A}\frac{b}{a}\right)t + \log\left(\frac{\tilde{A}}{A^{*}}\right) \\
&\quad - \frac{\theta}{1-\theta}\left(\frac{\tilde{A}}{a} - 1\right)\log\left(\frac{\frac{a}{b} + \left((K^{*})^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{(K^{*})^{1-\theta}}\right)
\end{aligned}$$

- $t > \tau$.

$$\begin{aligned}
\int_{t-\tau}^{t} g(u)\,du &= \rho\tau - \int_{t-\tau}^{t} r(u)\,du \\
&= \rho\tau - \int_{t-\tau}^{t} d\left[\frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{a}{b}e^{(1-\theta)bu} + (K^*)^{1-\theta} - \frac{a}{b}\right) - \delta u\right] \\
&= (\rho+\delta)\,\tau - \frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{\frac{a}{b}e^{(1-\theta)bt} + (K^*)^{1-\theta} - \frac{a}{b}}{\frac{a}{b}e^{(1-\theta)b(t-\tau)} + (K^*)^{1-\theta} - \frac{a}{b}}\right) \\
&= (\rho+\delta)\,\tau \\
&\quad - \frac{\theta\tilde{A}}{(1-\theta)\,a}\left((1-\theta)\,b\tau + \log\left(\frac{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}}\right)\right) \\
&= \left(\rho+\delta-\theta\tilde{A}\frac{b}{a}\right)\tau - \frac{\theta\tilde{A}}{(1-\theta)\,a}\log\left(\frac{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}}\right) \\
-\log\left(\frac{w(t-\tau)}{w_t}\right) &= \frac{\theta}{1-\theta}\log\left(\frac{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}}\right)
\end{aligned}$$

Consumption gap is thus

$$\begin{aligned}
-\log\left(\frac{c_{t-\tau,t}}{c_{t;h}}\right) &= -\log\left(\frac{w(t-\tau)}{w_t}\right) + \int_{t-\tau}^{t} g(u)\,du \\
&= \left(\rho+\delta-\theta\tilde{A}\frac{b}{a}\right)\tau - \frac{\theta}{1-\theta}\left(\frac{\tilde{A}}{a} - 1\right)\log\left(\frac{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)bt}}{\frac{a}{b} + \left((K^*)^{1-\theta} - \frac{a}{b}\right)e^{-(1-\theta)b(t-\tau)}}\right)
\end{aligned}$$

C Computational Details for the Figures

This section documents the computational details for the figures. We assume log utility when creating the figures.

- Figure 3: This figure plots the transitional dynamics of the aggregate variables, after a small permanent increase in productivity. The time path of these variables are computed in Section 5. The steady state values come from Section 3.

- Figure 4: This figure plots the transitional dynamics of consumption inequality, after a small permanent increase in productivity. We define the consumption gap, wage gap and discounting gap in Lemma 7 of Section 5.1. We characterize the time path of individual consumption in Section 4.2, and compute the the time path of the equilibrium wage and interest rate in Section 5.

- Figure 5: This figure plots consumption inequality as a function of population quantiles at different points in time along the transitional path. The shock is a small permanent increase in productivity. We create the figure in the following steps: at each time t,

  1. A low income household i is characterized by the time since he last had high income, which is denoted as τ_i. We create an equally spaced vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_n)$, which represents the cross section of low income households at time t.
  2. For each household i, we compute the corresponding quantile in the population ranked by consumption level, i.e. $P(\tau_i)$, using the definition of $P(\tau)$ in Section 5.1.
  3. For each household i, we compute his consumption ratio and consumption gap based on the individual consumption contracts in Section 4.2, and compute the wage gap and discounting gap (defined in Lemma 7 of Section 5.1) using the time path of the equilibrium wage and interest rate in Section 5.
  4. In each of the subfigure, we plot the population quantiles of these households (obtained from Step 2) on the x-axis, and one of the variables from Step 3 on the y-axis.

- Figure 6: This figure plots the CDF of the consumption ratio between low income household and high income household at different points in time along the transitional path. The shock is a small permanent increase in productivity. We create the figure in the following steps: at each time t,

  1. A low income household i is characterized by the time since he last had high income, which is denoted as τ_i. We create an equally spaced vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_n)$, which represents the cross section of low income households at time t.
  2. For each household i, we compute his consumption ratio based on the individual consumption contracts in Section 4.2.

3. For each household i, we compute the corresponding quantile in the population ranked by consumption level, i.e. $P(\tau_i)$, using the definition of $P(\tau)$ in Section 5.1.

4. Finally, we plot the consumption ratio of these households (obtained from Step 2) on the x-axis, and the corresponding population quantiles from Step 3 (which is also the CDF) on the y-axis.

- Figure 7: This figure plots the Lorenz curve at different points in time along the transitional path. The shock is a small permanent increase in productivity. We create the figure in the following steps: at each time t,

  1. A low income household i is characterized by the time since he last had high income, which is denoted as τ_i. We create an equally spaced vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_n)$, which represents the cross section of low income households at time t.

  2. For each household i, we compute the corresponding quantile in the population ranked by consumption level, i.e. $P(\tau_i)$, using the definition of $P(\tau)$ in Section 5.1.

  3. For each household i, we compute the cumulative consumption share (as a fraction of aggregate consumption) for those households with lower consumption level than i. The individual consumption level is computed under the individual consumption contracts in Section 4.2.

  4. Finally, we plot the population quantiles of these households (obtained from Step 2) on the x-axis, and the cumulative consumption share from Step 3 on the y-axis.

- Figure 8: This figure plots the transitional dynamics of the aggregate variables, after a permanent decrease in productivity. The time path of these variables are computed in Section 5. The steady state values come from Section 3.

- Figure 9: This figure compares the aggregate variable dynamics under a small permanent increase in productivity v.s. a symmetric permanent decrease in productivity. The computational details for these two cases are the same as in Figure 3 and Figure 8, respectively.

- Figure 10: This figure plots the transitional dynamics of consumption inequality in response to permanent shocks to productivity (either a small permanent increase or a symmetric decrease). The computational details are the same as in Figure 4.

- Figure 11: This figure plots consumption inequality as a function of population quantiles at different points in time along the transitional path. The shock is either a small permanent increase or a symmetric permanent decrease in productivity. The computational details are the same as in Figure 5.

- Figure 12: This figure plots the CDF of the consumption ratio between low income household and high income household at different points in time along the transitional path. The shock is either a small permanent increase or a symmetric permanent decrease in productivity. The computational details are the same as in Figure 6.

- Figure 13: This figure plots the Lorenz curve at different points in time along the transitional path. The shock is either a small permanent increase or a symmetric permanent decrease in productivity. The computational details are the same as in Figure 7.

We conclude this section by describing the features of each figure in Table C.1.

Table C.1: Summary of figures

Item	Shock to A	x-axis	y-axis
Figure 3	(small) permanent $\uparrow$	time	aggregate variables
Figure 4	(small) permanent $\uparrow$	time	consumption inequality
Figure 5	(small) permanent $\uparrow$	population quantile	consumption inequality
Figure 6	(small) permanent $\uparrow$	consumption ratio	CDF
Figure 7	(small) permanent $\uparrow$	population quantile	cumulative consumption share
Figure 8	permanent $\downarrow$	time	aggregate variables
Figure 9	both permanent $\uparrow$ and $\downarrow$	time	aggregate variables
Figure 10	both permanent $\uparrow$ and $\downarrow$	time	consumption inequality
Figure 11	both permanent $\uparrow$ and $\downarrow$	population quantile	consumption inequality
Figure 12	both permanent $\uparrow$ and $\downarrow$	consumption ratio	CDF
Figure 13	both permanent $\uparrow$ and $\downarrow$	population quantile	cumulative consumption share